Informal Lecture Notes for

# ME 4812
# Fluid Power Control

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# Contents

# 1 FUNDAMENTALS

## 1.1 Introduction

Hydraulic components are used primarily as actuation elements of power–control systems. Some of the advantages of hydraulic systems are:

1. High–pressure hydraulic power can be generated efficiently, with pump efficiencies of 92 percent common.
2. Hydraulic components are comparatively light in weight compared with equivalent mechanical and electrical components because the highly stressed structures of the hydraulic system make very efficient use of structural material. Hydraulic pumps and motors with power density less than 1 lb/hp are common. This light weight is made possible by the high pressures available from commercial pumps. Hydraulic systems operating at 3000 psi and higher are quite common.
3. The hydraulic fluid acts as a heat exchanger, this results in smaller and lighter components.
4. The hydraulic fluid acts as a lubricant, this results in longer component life.
5. The hydraulic actuator is extremely stiff compared with an equivalent pneumatic or electrical system. This means that the operating condition is maintained against load disturbances.
6. System response is very linear. Hydraulic actuators develop relatively large torques for small devices.
7. Hydraulic actuation offers the highest torque to inertia ratio in comparison with most mechanical, pneumatic, and electrical systems. This property, coupled with the incompressible nature of the medium, results in exceptionally fast response and high power output.

Some of the disadvantages of hydraulic systems are:

1. Most hydraulic systems use organic-based fluids which present serious fire and explosion hazards, particularly at high temperatures.
2. Difficulty of preventing leaks in normal usage.
3. Inevitable fluid contamination, which results in bad reliability and the need for constant maintenance.
4. They are difficult to design, fluid flow is not always easy to predict or analyze.
5. Hydraulic components are not desirable in low power control systems.

## 1.2 Hydraulic Fluids

1. Basic Properties:

By definition, a fluid is a medium that cannot withstand shear force. The density of the fluid is defined as the mass per unit volume. The specific gravity is weight per unit volume and the specific gravity $\sigma$ is the ratio of the density of the substance in question to that of water at 60° F. The petroleum industry uses a measure of relative density called "API gravity." API gravity in terms of specific gravity is,

$$\text{Degrees API} = \frac{141.5}{\sigma 60°F/60°F} - 131.5\,,$$

where $\sigma 60°F/60°F$ represents the specific gravity of the substance at 60°F relative to water at 60°F. The mass density of a fluid is a function of both pressure and temperature. It increases with increasing pressure and decreases with increasing temperature. At a given temperature, a good approximation is

$$\rho = \rho_0(1 + aP - bP^2)\,,$$

with typical values for hydraulic oil

$$\begin{aligned} a &= 4.38 \times 10^{-6}\ \text{in}^2/\text{lb} \\ b &= 5.65 \times 10^{-11}\ \text{in}^4/\text{lb}^2\,. \end{aligned}$$

At a constant pressure:

$$\rho = \rho_0[1 - \alpha(T - T_0)]\,,$$

where

$$\alpha = \text{cubical expansion coefficient}\,.$$

This linear approximation is accurate within 0.5 percent for most hydraulic fluids over temperature ranges of 500° F. For small changes in both $P$ and $T$:

$$\begin{aligned} \rho &= \rho_0 + \left(\frac{\partial \rho}{\partial P}\right)_T (P - P_0) + \left(\frac{\partial \rho}{\partial T}\right)_P (T - T_0) \\ &= \rho_0 \left[1 + \frac{1}{\beta}(P - P_0) - \alpha(T - T_0)\right]\,, \end{aligned}$$

the linearized equation of state for a liquid, where

$$\begin{aligned} \alpha &= -\frac{1}{\rho_0}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{1}{V_0}\left(\frac{\partial V}{\partial T}\right)_P \\ \beta &= -V_0\left(\frac{\partial P}{\partial V}\right)_T = \rho_0\left(\frac{\partial P}{\partial \rho}\right)_T \end{aligned}$$

Occasionally, $\alpha$ and $\beta$ are defined with respect to the instantaneous values of volume and density,

$$\alpha = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_P = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_P$$

$$\beta = -V\left(\frac{\partial P}{\partial V}\right)_T = \rho\left(\frac{\partial P}{\partial \rho}\right)_T$$

$\beta$ is called the bulk modulus (the reciprocal of $\beta$ is the compressibility) and is always positive. In hydraulic systems the bulk modulus of the pure fluid can be drastically reduced; e.g., from entrained air. In terms of $a$ and $b$, the fluid's bulk modulus is given by

$$\beta = \frac{1 + aP - bP^2}{a - 2bP}.$$

2. <u>Viscosity</u>:

Fluids cannot withstand shear: any shear force will result in a finite shear rate. A Newtonian fluid is one for which the shear rate is proportional to the shear stress. The constant of proportionality is called the absolute viscosity, $\mu$,

$$\mu = \frac{\tau}{(du/dx)};$$

where

$$\begin{aligned} \tau &= \text{shear stress} \\ du &= \text{change in velocity resulting from shear stress} \\ x &= \text{direction of shear stress} \end{aligned}$$

The kinematic viscosity is defined by

$$\nu = \frac{\mu}{\rho}.$$

The viscosity of fluids increases with pressure

$$\log_{10}\left(\frac{\mu}{\mu_0}\right) = cP;$$

and decreases markedly with temperature

$$\mu = \mu_0 e^{-\lambda(T - T_0)}.$$

For most petroleum products at room temperature,

$$c = 7 \times 10^{-4}\ \text{in}^2/\text{lb}.$$

Typical viscosity/bulk modulus–temperature curves are shown in Figure 1.

Since several units of viscosity are in use, they should be carefully defined:

- **Reyn.** A very large inconvenient unit in the English system, 1 Reyn = 1 lb s/in$^2$.

- **Centipoise (cP)** (metric system). One centipoise is the viscosity of a fluid such that a force of 1 dyne will give two parallel surfaces 1 cm$^2$ area, 1 cm apart, a velocity of $0.01$ cm/s. The centipoise is thus $0.01$ dyne $\cdot$ s/cm$^2$.

- **Centistoke (cSt).** This is a unit for kinematic viscosity and it corresponds to the centipoise divided by the density in consistent units. The centistoke is thus $0.01$ cm$^2$/s.

- **Saybolt Universal Seconds.** The Saybolt viscosimeter is commonly used to determine the viscosity of petroleum products. The time required for 60 mL of the sample to flow through a $0.176$ cm diameter and $1.225$ cm long tube is measured and designated SSU.

3. <u>Thermal Properties</u>:

Specific heat, $C_P$, is the amount of heat required to raise the temperature of a unit mass by 1 degree. For fluid at moderate temperatures $C_P \approx C_V$. Thermal conductivity is a measure of the rate of heat flow through an area for a temperature gradient in the direction of heat flow. For petroleum-base oils:

- specific heat

$$C_P = \frac{1}{\sqrt{\sigma}}(0.388 + 0.00045T)\,;$$

  where

$$\begin{aligned} C_P &= \text{specific heat, BTU/lb} \cdot {}^{o}\text{F} \\ \sigma &= \text{specific gravity at } 60^{o}\text{F} \\ T &= \text{temperature, } {}^{o}\text{F} \end{aligned}$$

- thermal conductivity

$$k = \frac{0.813}{\sigma}[1 - 0.0003(T - 32)]$$

  in Btu/h $\cdot$ ft$^2$ $\cdot$ $^{o}$F $\cdot$ in.

4. <u>Effective Bulk Modulus</u>:

The bulk modulus of a liquid can be substantially lowered by entrained air and/or mechanical compliance. Consider the fluid shown schematically in Figure 2, where the initial total volume of the container is the sum of the pure fluid and entrained gas volumes,

$$V_t = V_\ell + V_g\,.$$

After the piston moves to the left there is a decrease in the initial volume of

$$-\Delta V_t = -\Delta V_g - \Delta V_\ell + \Delta V_c\,;$$

where

$$
\begin{aligned}
-\Delta V_g &= \text{decrease in gas volume} \\
-\Delta V_\ell &= \text{decrease in liquid volume} \\
\Delta V_c &= \text{increase in container volume}
\end{aligned}
$$

The effective bulk modulus will be defined as

$$\frac{1}{\beta_e} = -\frac{\Delta V_t}{V_t \Delta P}$$

or

$$\frac{1}{\beta_e} = \frac{V_g}{V_t}\left(-\frac{\Delta V_g}{V_g \Delta P}\right) + \frac{V_\ell}{V_t}\left(-\frac{\Delta V_\ell}{V_\ell \Delta P}\right) + \left(\frac{\Delta V_c}{V_t \Delta P}\right):$$

Since

$$
\begin{aligned}
\beta_\ell &= -\frac{V_\ell \Delta P}{\Delta V_\ell} \\
\beta_g &= -\frac{V_g \Delta P}{\Delta V_g}
\end{aligned}
$$

we have

$$\frac{1}{\beta_e} = \frac{V_g}{V_t}\left(\frac{1}{\beta_g}\right) + \frac{V_\ell}{V_t}\left(\frac{1}{\beta_\ell}\right) + \frac{1}{\beta_c};$$

where

$$\beta_c = \frac{V_t \Delta P}{\Delta V_c}$$

is some kind of bulk modulus of the container with respect to the total volume. $\beta_c$ can also be written as

$$\frac{1}{\beta_e} = \frac{1}{\beta_c} + \frac{1}{\beta_\ell} + \frac{V_g}{V_t}\left(\frac{1}{\beta_g} - \frac{1}{\beta_\ell}\right)$$

and since

$$\beta_\ell \gg \beta_g$$

we get

$$\frac{1}{\beta_e} = \frac{1}{\beta_c} + \frac{1}{\beta_\ell} + \frac{V_g}{V_t} \cdot \frac{1}{\beta_g} > \frac{1}{\beta_\ell}$$

or

$$\beta_e < \beta_\ell:$$

The most difficult task in applying this formula is in determining the bulk modulus of containers due to mechanical compliance. The major source of mechanical compliance is the hydraulic lines connecting valves and pumps to actuators. For a thin-walled steel cylinder

$$\beta_c = \frac{TE}{D}$$

where

$$T = \text{wall thickness}$$
$$E = \text{modulus of elasticity}$$
$$D = \text{diameter}$$

For a thick cylinder

$$\beta_c = \frac{E}{2.5}.$$

Bulk modulus for a gas is

$$\beta_g = 1.4P$$

where $P$ is the pressure.

Example: For a petroleum base fluid $\beta_\ell = 2.2 \times 10^5$ psi. Suppose that the fluid is inside a steel pipe at pressure 500 psi and contains 1% (by volume) of entrapped air. Let $D = 6T$; what is its $\beta_e$?

$$\beta_c = \frac{TE}{6T} = \frac{1}{6} \times 30 \times 10^6 = 5 \times 10^6 \text{ psi}$$
$$\beta_g = 1.4 \times 500 = 700 \text{ psi}$$
$$\frac{1}{\beta_e} = \frac{1}{5 \times 10^6} + \frac{1}{2.2 \times 10^5} + \frac{0.01}{700} = 1.904 \times 10^{-5} \Rightarrow \beta_e = 52600 \text{ psi}$$

In the absence of entrapped air we would get $\beta_e = 210000$ psi; in other words 1% air causes $\beta_e$ to decrease by a factor of 4. If the pressure $P$ were 1000 psi, then $\beta_e = 84100$ psi. This is an advantage that high pressure systems offer.

> Because entrained air reduces the bulk modulus, the natural frequency of hydraulic actuators in servo systems may be lowered to such an extent that system instability occurs.

5. Chemical Properties:

The most important chemical properties are:

- Thermal: Some hydraulic fluids when heated to high temperature decompose to form gaseous, liquid, or solid products.
- Oxidative: Reaction of hydraulic fluids with oxygen.
- Hydrolytic: Reaction of hydraulic fluids with water.

With regards to fire safety:

- Flash point: Temperature at which vapors are formed and cause a transient flame under the application of a test flame.

- Fire point: Temperature at which transient flame is self sustaining for 5 seconds, usually about 50 degrees F higher than flash point.
- Autogenous ignition: Considerably higher than fire point; temperature at which a liquid droplet ignites upon contact with heated air.

6. <u>Surface Properties</u>:

Two main types:

- Foaming: Emulsion of gas bubbles in a liquid. Antifoaming additives are frequently added in the hydraulic fluid.
- Boundary lubrication: It relates to physicochemical relations occurring in thin films at the fluid–metal interface.

7. <u>Choice of Hydraulic Fluid</u>:

System performance, both steady state and transient, is affected by fluid properties as follows:

- Viscosity: Pipe flow, lubrication, leakage, system efficiency.
- Density: Orifice flow, acoustic effects, system efficiency.
- Compressibility: Transmission characteristics, stability and response of closed–loop control systems.
- Specific heat and thermal conductivity: Combined with viscosity and density affect temperature rise and heat dissipation.
- Vapor pressure: Affects cavitation effects.

Hydraulic system life and reliability are closely associated with such fluid properties as:

- Boundary lubrication affects wear in pumps and motors.
- Thermal stability: Poor performance results in high gas emission.
- Compatibility; i.e., the property of the fluid to be affected or to affect surrounding metallic and nonmetallic materials: Poor performance may result in side effects such as seal deterioration.

## 1.3 Fundamentals of Hydraulic Flow

1. Introduction:

In fluid flows there are four types of equations that need to be written:

1. Conservation of momentum or Newton's law requires that the net rate of outflow of momentum in a specific direction x plus the rate at which momentum accumulates within the control volume be equal to the force applied to the control volume in the x direction,

$$\sum F_x = \frac{d}{dt}\int_V \rho U \, dV + \int_A \rho u U_n \, dA \; :$$

In differential form the same principle is expressed by the Navier–Stokes equations as,

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho X - \frac{\partial P}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

and similarly for y, z directions.

2. Conservation of mass or continuity requires that the rate of mass flow into a control volume equal the rate of mass flow out plus the rate at which mass accumulates within the control volume,

$$\int_A \rho U_n \, dA + \frac{d}{dt}\int_V \rho \, dV = 0 \; ;$$

or

$$\sum W_{in} - \sum W_{out} = g\frac{d(\rho V_0)}{dt} \; :$$

3. Conservation of energy requires that the increase in internal energy of a system be equal to the work done on the system plus the heat added to the system,

$$\frac{dE}{dt} = \frac{dQ_n}{dt} + \frac{dW_x}{dt} + \int_A \left(\frac{P}{\rho} + e\right)\rho U_n \, dA \; ;$$

where

| | | |
|---|---|---|
| $Q_n$ | = | heat flow to the control volume |
| $W_x$ | = | shaft and shear work done on the system |
| $E$ | = | total internal energy of fluid inside the control volume |
| $e$ | = | total internal energy per unit of mass; |
| | | $e = u + gZ + \frac{U^2}{2}$ where |
| | | $u$ = intrinsic internal energy per unit mass |
| | | $Z$ = height above a reference point |
| $P$ | = | pressure on an element of area at the surface of the control volume |

4. Constitutive relations or equations of state express the density and viscosity as functions of temperature and pressure:

$$
\begin{aligned}
\rho &= \rho(P, T) \\
\mu &= \mu(P, T).
\end{aligned}
$$

A very important quantity of a flow is the Reynolds number, which represents the ratio of inertia to viscous forces and is defined by

$$Re = \frac{\rho U a}{\mu} = \frac{U a}{\nu},$$

where

$$
\begin{aligned}
U &= \text{average or reference velocity of flow} \\
a &= \text{a characteristic length (e.g., a diameter or length)}
\end{aligned}
$$

For low $Re$, the flow is dominated by viscosity, we refer to this as laminar flow. For high $Re$, the flow is dominated by inertia and is referred to as turbulent flow.

For one-dimensional, steady, incompressible, frictionless flow with no body forces, the Navier-Stokes equations simplify to

$$\frac{u^2}{2g} + \frac{P}{\rho g} + Z = \text{const}$$

This is Bernoulli's equation and is applicable along a stream line of potential flow.

2. Flow in Pipes:

It can be either laminar or turbulent. Reynolds number is based on pipe diameter $D$,

$$Re = \frac{U D}{\nu}.$$

In general,

$$
\begin{aligned}
Re < 2000 &\Rightarrow \text{laminar flow} \\
Re > 4000 &\Rightarrow \text{turbulent flow}.
\end{aligned}
$$

The pressure drop for laminar flow in circular cross sections is given by,

$$\frac{P_1 - P_2}{L} = \frac{128\mu}{\pi D^4} Q.$$

For turbulent flow,

$$\frac{P_1 - P_2}{L} = 0.242 \frac{\mu^{0.25} \rho^{0.75}}{D^{4.75}} Q^{1.75};$$

where

$$
\begin{aligned}
P_1 - P_2 &= \text{pressure drop, psi} \\
Q &= \text{volume flow rate, in}^3/\text{sec} \\
L &= \text{pipe length, in} \\
D &= \text{pipe inside diameter, in} \\
\rho &= \text{fluid mass density, lb}\cdot\text{sec}^2/\text{in}^4 \\
\mu &= \text{fluid viscosity, lb}\cdot\text{sec/in}^2
\end{aligned}
$$

For laminar flow in noncircular cross sections see Fig. 3.

For turbulent flows in noncircular cross sections we can use the above equation, but with the hydraulic diameter $D_h$ instead of $D$:

$$D_h = \frac{4A}{S}$$

where

$$
\begin{aligned}
A &= \text{flow section area} \\
S &= \text{flow section perimeter}
\end{aligned}
$$

For cicrular sections, the above formula produces $D_h = D$ as it should.

3. <u>Flow through Orifices</u>:

The design of valves for control and regulation purposes and the design of pumps and motors require the analysis of flow through rounded and sarp–edged orifices. Consider the flow through the orifice, schematically depicted in Figure 4. We denote

$A_0$ : orifice area

$A_2$ : stream area at the point where the jet area is minimum.

We define the contraction coefficient of the orifice,

$$C_c = \frac{A_2}{A_0}.$$

Using Bernoulli's equation between points 1 and 2,

$$P_1 + \frac{1}{2}\rho U_1^2 = P_2 + \frac{1}{2}\rho U_2^2;$$

and the continuity equation,

$$Q = A_1 U_1 = A_2 U_2;$$

we get the expression for the flow rate

$$Q = C_d A_0 \sqrt{\frac{2}{\rho}(P_1 - P_2)};$$

where the discharge coefficient $C_d$ is defined by

$$C_d = \frac{C_v C_c}{\sqrt{1 - C_c^2 (A_0/A_1)^2}} \, .$$

The coefficient $C_v$ is called the velocity coefficient, and is an empirical factor introduced to account for the fact that, because of viscous friction, the jet velocity is always less that the theoretical value. $C_v$ is normally around $0.98$ and can be set equal to one in most practical applications. Since $A_0 \ll A_1$ it follows then that $C_d \approx C_c$. The discharge coefficient is difficult to compute, but a good approximation for most orifices is

$$C_d \approx 0.6 \, .$$

Since $C_d \sqrt{2/\rho} \approx 100\ \mathrm{in}^2 \sqrt{\mathrm{lb}}$–sec, the orifice equation can be written as

$$Q = 100 A_0 \sqrt{P_1 - P_2} \, ;$$

where pressures are in psi, orifice area is in $\mathrm{in}^2$, and volumetric flow rate is in $\mathrm{in}^3/\mathrm{sec}$. The above approximation is good for orifices of zero length; for an orifice with non–zero length the discharge coefficient is usually less, see Figure 5.

The above expressions are valid for turbulent flow, which is normally the case for orifice flow. For laminar flow; i.e., very low Reynolds number, the discharge coefficient is smaller and the flow rate is proportional to the pressure difference instead of the square root of the pressure difference as in the case of turbulent flow. This is, for our purposes, the most important difference between laminar and turbulent flows.

# 2 HYDRAULIC ACTUATORS

## 2.1 Introduction

Hydraulic actuators are used to convert fluid to mechanical energy and vice versa,

$$\begin{array}{c} \text{Fluid} \\ \text{Energy} \end{array} \; \underset{\text{pump}}{\overset{\text{motor}}{\rightleftarrows}} \; \begin{array}{c} \text{Mechanical} \\ \text{Energy} \end{array}$$

There are two main types of hydraulic actuators:

- Hydrodynamic (turbine): Continuous flow from inlet to outlet. Low pressure machines with high volume output. They are used primarily for auxiliary functions and not control purposes.

- Positive displacement: Fluid passes through the inlet into a chamber which expands in volume and fills with fluid. The volume expansion causes shaft rotation in the case of a motor or vice versa for a pump. The volume of trapped fluid is transported to the outlet side where it is discharged. They are extensively used in control systems and they can generate relatively high pressures at relatively low flows. Unlike hydrodynamic pumps, which can tolerate fluids with considerable contaminant content, positive displacement pumps require clean fluids of good lubricity and adequate viscosity.

There are three main types of positive displacement actuators:

- Gear devices are used extensively for jet fuels, lubricating oils, and other applications where pressures up to 1500 $lb/in^2$ suffice. Gear pumps are fixed displacement pumps; i.e., delivery per revolution cannot be changed over large ranges with good retention of efficiency.
- Vane actuators find broad use in such applications as roadworking machinery, machine tool application, and many other uses where pressures do not exceed 2000 $lb/in^2$. Variable displacement vane pumps are available but pressures rarely exceed 1500 $lb/in^2$.
- High pressure generation of fluid power in the 3000 to 5000 $lb/in^2$ range can be accomplished by the piston actuator. Either axial piston or radial piston actuators, fixed and variable displacement, are available.

Figures 6 and 7 show schematically typical axial piston actuators. These units are quite compact and provide a high power volume ratio. They are capable of operating in configurations such that the angle between the drive shaft and the cylinder block is adjustable, see Fig. 7. Adjustment of this angle can cause the pump displacement to vary continuously from zero to maximum.

## 2.2 Energy Considerations — Ideal Analysis

Consider a piston of cross section area $A$. If the fluid pressure on either side of the piston is $P_1$ and $P_2$, the total force on the piston is,

$$F = A(P_1 - P_2):$$

If we denote the load pressure drop by

$$P_L = P_1 - P_2;$$

the work done by the piston during a translational motion $\Delta S$ is

$$\Delta W = AP_L \Delta S = P_L \Delta V;$$

where $\Delta V$ is the volume swept by the piston during the motion. The fluid power is

$$P_{in} = \frac{\Delta W}{\Delta t} = P_L \frac{\Delta V}{\Delta t} = P_L Q_L;$$

where we have denoted

$$Q_L = \frac{\Delta V}{\Delta t}$$

as the load flow rate.

The output mechanical power is

$$P_{out} = \frac{\Delta W}{\Delta t} = \frac{T_g \Delta \theta}{\Delta t} = T_g \dot{\theta}_m \; ,$$

where

$$\begin{aligned} T_g &= \text{generated torque} \\ \dot{\theta}_m &= \text{shaft speed of motor}. \end{aligned}$$

Assuming no losses; i.e., ideal analysis,

$$P_{in} = P_{out}$$

or

$$P_L Q_L = T_g \dot{\theta}_m \; ,$$

If we define

$$D_m = \frac{Q_L}{\dot{\theta}_m}$$

as the displacement of the actuator, we get

$$T_g = D_m P_L \; .$$

We can see that the displacement is by definition the flow rate per unit motion.

<u>Remark:</u> This is true for a rotating device. For a piston type actuating device the piston area is the parameter analogous to the displacement of a rotary device. Since,

$$\begin{aligned} F_g \dot{x}_p &= P_L Q_L \\ \frac{Q_L}{\dot{x}_p} &= A_p \end{aligned}$$

we get

$$F_g = A_p P_L \; .$$

## 2.3 Real Motor Analysis

There are two primary sources of losses in hydraulic devices: leakage flows and friction. Therefore, we can identify two types of efficiency: volumetric efficiency and torque efficiency. We study each one separately.

1. <u>Volumetric efficiency:</u> Consider steady state; i.e., compressibility is not an issue. Figures 8 and 9 show schematically the flows in an axial motor. If the displacement of the motor is $D_m$ and the shaft speed $\theta_m$, the ideal flow through the motor would be

$$Q_L' = D_m \theta_m \; .$$

Continuity gives

$$\begin{aligned} Q_1 &= Q_{im} + Q_{em1} + Q_L' \\ Q_2 &= Q_L' - Q_{em2} + Q_{im} \; , \end{aligned}$$

where $Q_{im}$ is an internal leakage flow and $Q_{em1}$ ($Q_{em2}$) is an external leakage flow at the supply (return) line. Leakage flows occur at sufficiently low Reynolds numbers so that they are modeled as laminar flows. Therefore, the flow rate will be proportional to pressure difference:

$$\begin{aligned} Q_{im} &= C_{im}(P_1 - P_2) \\ Q_{em1} &= C_{em1}(P_1 - P_0) \\ Q_{em2} &= C_{em2}(P_2 - P_0) \; , \end{aligned}$$

where

$$\begin{aligned} C_{im} &= \text{internal leakage coefficient} \\ C_{em} &= \text{external leakage coefficient} \; . \end{aligned}$$

Without loss of generality we can assume that all pressures are gage pressures,

$$P_0 = 0 \; ,$$

and

$$C_{em1} = C_{em2} \; .$$

Therefore, we get

$$Q_1 + Q_2 = 2Q_L' + Q_{em1} - Q_{em2} + 2Q_{im} \; ,$$

or

$$Q_L = D_m \theta_m + \left( C_{im} + \frac{C_{em}}{2} \right) P_L \; ,$$

where we have denoted

$$Q_L = \frac{Q_1 + Q_2}{2} \; ,$$

the load flow, which is an average of the flows in the two motor lines.

The volumetric efficiency is defined as the ratio of flow which results in motor speed (the ideal flow) to the flow supplied to the motor

$$\eta_v = \frac{D_m \theta_m}{Q_1} \; .$$

Since

$$Q_1 = D_m \dot\theta_m + (C_{em} + C_i)P_1\,;$$

with

$$P_2 = P_0 = 0$$

we get

$$\eta_v = \frac{1}{1 + \frac{C_{im} + C_{em}}{D_m \dot\theta_m} P_1}\,.$$

We can define the slip flow by

$$Q_s = (C_{im} + C_{em})P_1\,.$$

Since the slip flow is laminar, it is inversely proportional to viscosity

$$Q_s = C_s \frac{D_m}{\mu} P_1$$

where

$$C_s = \frac{\mu}{D_m}(C_{em} + C_{im})\,;$$

is the coefficient of slip. Therefore, the volumetric efficiency can also be written as

$$\eta_v = \left(1 + \frac{C_s P_1}{\mu \dot\theta_m}\right)^{-1}.$$

2. <u>Torque efficiency:</u> The ideally generated torque is

$$T_g = D_m(P_1 - P_2)\,.$$

In reality, however,

$$T_g = T_d + T_f + T_c + T_L\,;$$

where

$$\begin{aligned}
T_d &= \text{loss due to fluid friction (damping)}\\
T_f &= \text{loss due to internal (mechanical) friction}\\
T_c &= \text{loss due to seal friction}\\
T_L &= \text{what's left over – load torque.}
\end{aligned}$$

We study each one separately.

$T_d$ is the torque required to shear the fluid in the small tolerances between mechanical elements in relative motion; it is proportional to motor speed,

$$T_d = B_m \dot\theta_m = C_d D_m \mu \dot\theta_m\,;$$

where

$$\begin{aligned}
B_m &= C_d D_m \mu = \text{viscous damping coefficient}\\
C_d &= \text{dimensionless damping coefficient}
\end{aligned}$$

To see why $T_d$ is proportional to $\dot{\theta}_m$, consider:

$$T_d \approx \tau L^2 \cdot L, \; \tau = \mu \frac{U}{L}, \; U \approx \dot{\theta}_m L \; \Rightarrow \; T_d \approx \mu \dot{\theta}_m L^3, \; L^3 \approx D_m .$$

Since $T_d$ is proportional to $\dot{\theta}_m$ it represents a kind of damping torque.

$T_f$ is the torque lost during transformation of piston motion into rotary shaft motion, it is due to mechanical friction.

To establish a relationship for $T_f$, consider

$$F_f \approx \mu_s F, \; F \approx (P_1 + P_2)L^2, \; T_f \approx L F_f \approx \mu_s (P_1 + P_2) L^3 \approx \mu_s D_m (P_1 + P_2) .$$

Therefore, $T_f$ is proportional to $D_m(P_1 + P_2)$, and since it must reverse its direction with motor speed, we can write

$$T_f = \frac{\dot{\theta}_m}{|\dot{\theta}_m|} C_f D_m (P_1 + P_2),$$

where

$$C_f = \text{friction coefficient}$$
$$C_{fs} = \text{static friction coefficient}$$

and steady-state performance is assumed. Typical curves illustrating the transition from starting to running friction are shown in Figure 10.

$T_c$ is a constant torque loss, it reverses direction with speed $[(\dot{\theta}_m / |\dot{\theta}_m|) T_c]$ just like $T_f$, and is usually neglected.

Assuming positive motor speed $\dot{\theta}_m$ and substituting, we get

$$P_L D_m = C_d D_m \mu \dot{\theta}_m + C_f D_m (P_1 + P_2) + T_c + T_L .$$

The torque or mechanical efficiency is defined by

$$\eta_t = \frac{\text{(available torque)}}{\text{(generated torque)}} = \frac{T_L}{P_L D_m} .$$

If we assume $P_2 = 0$ and neglect $T_c$, we get

$$\eta_t = \frac{P_1 D_m - C_d D_m \mu \dot{\theta}_m - C_f D_m P_1}{P_1 D_m} = 1 - \frac{C_d \mu \dot{\theta}_m}{P_1} - C_f .$$

The over-all efficiency is

$$\eta_{oa} = \frac{P_{out}}{P_{in}} = \frac{T_L \dot{\theta}_m}{Q_1 P_1} = \frac{T_L}{D_m P_1} \cdot \frac{D_m \dot{\theta}_m}{Q_1} = \eta_t \eta_v$$

or

$$\eta_{oa} = \frac{1 - (C_d \mu \theta_m / P_1) - C_f}{1 + (C_s P_1 / \mu \theta_m)} .$$

Therefore, static performance of a motor with zero return pressure can be defined by the parameters $C_s$, $C_d$, $C_f$, and the dimensionless quantity $\mu \theta_m / P_1$. We can see that, as Figure 11 demonstrates, as the nondimensional motor speed $\mu \theta_m / P_1$ is increased, the volumetric efficiency $\eta_v$ also increases, while the torque efficiency $\eta_t$ is decreased. The overall efficiency reaches an optimum value for a certain motor speed.

Remark: The above expressions are true for motor. For a pump, an analogous procedure shows that

$$\eta_{oa} = \frac{1 - (C_s P_1 / \mu N_p)}{1 + (C_d \mu N_p / P_1) + C_f} ;$$

where the pump speed is denoted by $N_p$.

## 2.4 Experimental Realizations

We want to determine the basic motor performance parameters from a series of tests. The load torque is

$$T_L = P_L D_m - (T_d + T_f + T_c)$$

or

$$T_L = P_L D_m - \left[ C_d \mu D_m \theta_m + C_f D_m (P_1 + P_2) + T_c \right] .$$

"Controllable" parameters are $P_1, P_2, P_L = P_1 - P_2, \theta_m, T_L$, and $P_1 + P_2 = P_L + 2P_2$. If we keep $P_2$ and $\theta_m$ constant, we get

$$\frac{\partial T_L}{\partial P_L} = D_m (1 - C_f) .$$

As figure 12 shows, $(T_L; P_L)$ is the graph of a straight line and from the slope of this straight line we can get the quantity $D_m(1 - C_f)$.

In order to determine friction characteristics we unload the motor ($T_L = 0$) and measure pressure difference at various return pressure levels,

$$D_m P_L = C_d \mu D_m \theta_m + P_L C_f D_m + 2P_2 C_f D_m + T_c ;$$

or

$$P_L = \frac{1}{D_m (1 - C_f)} \left( C_d \mu D_m \theta_m + 2P_2 C_f D_m + T_c \right) .$$

Therefore, with $\theta_m$ constant,

$$\frac{\partial P_L}{\partial P_2} = \frac{2C_f}{1 - C_f}$$

and the starting value is at

$$\frac{T_c}{D_m (1 - C_f)} .$$

We can get then $C_f$ from this graph, Figure 13, and then obtain $D_m$ from the previous experiment, Figure 12.

To measure torque losses that depend on speed, we set again $T_L = 0$ and $P_2 = 0$ (or constant), and measure $P_L$ versus $\theta_m$, see Figure 14,

$$\frac{\partial P_L}{\partial \theta_m} = \frac{\partial P_1}{\partial \theta_m} = \frac{C_d \mu}{1 - C_f}:$$

The starting value is at

$$\frac{T_c}{D_m(1 - C_f)}$$

and from the slope of the curve and the previous results we can get $C_d$.

Motor leakage characteristics can be determined by locking the motor shaft and setting $P_2 = 0$. Then apply pressure $P_1$ and measure the flows in the return and drain lines. The return line flow is the internal leakage and the drain line flow is the external leakage, see Figure 15. The slopes of these two curves versus $P_1$ give the desired coefficients $C_{im}$ and $C_{em}$.

## 2.5 Typical Hydraulic Pump Constants

Typical values for usual pumps are:

| Unit | $D_m$ (in$^3$/rev) | $C_d$ | $C_s$ | $C_f$ | $T_c$ |
|---|---|---|---|---|---|
| Piston pump | 3.6 | $16.8 \times 10^4$ | $0.15 \times 10^{-7}$ | 0.045 | 0 |
| Vane pump | 2.8 | $7.3 \times 10^4$ | $0.47 \times 10^{-7}$ | 0.212 | 0 |
| Gear pump | 2.9 | $10.2 \times 10^4$ | $0.48 \times 10^{-7}$ | 0.179 | 0 |

# 3 HYDRAULIC CONTROL VALVES

## 3.1 Introduction

Hydraulic control valves use mechanical motion to control fluid power. By throttling the fluid power in a single or multiple–orifice valve, they provide control by predictable flow restrictions. There are three main types of hydraulic valves:

- spool valves
- flapper valves
- jet pipe valves

as is schematically shown in Figure 16.

Spool valves are classified according to the number of lands and the number of ways the flow can enter and leave the valve, see Fig. 16. A three–way valve is the simplest configuration which permits load reversal, a four–way valve is the most common in practice. If the land width is less than the port in valve sleeve, it is called an open center valve or underlapped. Otherwise it is called critical center or zero lapped valve, and closed center or overlapped valve.

The single most important characteristic of a valve is the flow gain, which is the slope of the load flow $Q_L$ vs. spool stroke $x_v$ curve. Typical flow gain curves are shown in Figure 17. Most four–way valves are manufactured with a critical center because of the desirable feature of the linear flow gain. Closed center valves are not desirable because of the "dead–band" nonlinearity which can cause stability problems. Open center valves exhibit "bi–linear" gain characteristics: the gain at nonzero set points is lower which results in larger steady state errors and decreased bandwidth or control system responsiveness. This is an undesirable feature.

Spool valves require close and matching tolerances, therefore such valves are relatively expensive and sensitive to fluid contamination. The required tolerances for flapper valves are not as strict, although the relatively large leakage flows of flappers limit their application to low power levels. Flapper valves are used almost exclusively as the first stage valve in two–stage servovalves. Jet pipe valves are not used as often because of their larger leakage flows and slower response times.

## 3.2 Flow Analysis

Consider the typical four–way spool valve shown in Figure 18. Suppose that the spool is given a positive displacement from the null or neutral position, that is, the position $x_v = 0$, which is chosen to be the symmetrical position of the spool in its sleeve.

Assuming steady state, we can neglect compressibility, and we denote

$$\begin{aligned} P_S &= \text{supply pressure} \\ P_0 &= \text{return pressure} \\ P_L &= P_1 - P_2 . \end{aligned}$$

All flows, including leakage bypass flows, can be assumed to be orifice flows:

$$\begin{aligned} Q_1 &= K_1 \sqrt{P_S - P_1} \\ Q_2 &= K_2 \sqrt{P_S - P_2} \\ Q_3 &= K_1 \sqrt{P_2 - P_0} \\ Q_4 &= K_2 \sqrt{P_1 - P_0} \end{aligned}$$

where

$$K_i = C_d A_i \sqrt{\frac{2}{\rho}} \qquad i = 1, \ldots, 4 .$$

Assuming matched orifices we have

$$\begin{aligned} A_1 &= A_3; \\ A_2 &= A_4. \end{aligned}$$

If the orifices are also symmetrical,

$$\begin{aligned} A_1(x_v) &= A_2(-x_v) \\ A_3(x_v) &= A_4(-x_v). \end{aligned}$$

Therefore,

$$A_1(0) = A_2(0) = A_3(0) = A_4(0) = A_0.$$

Assuming no external leakage at the load, continuity gives

$$\begin{aligned} Q_L &= Q_1 - Q_4; \\ Q_L &= Q_3 - Q_2. \end{aligned}$$

Therefore,

$$Q_1 - Q_3 = Q_4 - Q_2.$$

Algebraic manipulation produces,

$$\begin{aligned} & Q_1^2 - Q_3^2 = K_1^2[(P_S - P_1) - (P_2 - P_0)] = +K_1^2(P_S + P_0 - P_1 - P_2) \\ & Q_4^2 - Q_2^2 = K_2^2[(P_1 - P_0) - (P_S - P_2)] = -K_2^2(P_S + P_0 - P_1 - P_2) \\ \Rightarrow\; & K_2^2(Q_1 - Q_3)(Q_1 + Q_3) = -K_1^2(Q_4 - Q_2)(Q_4 + Q_2) \\ \Rightarrow\; & (Q_1 - Q_3)[K_2^2(Q_1 + Q_3) + K_1^2(Q_4 + Q_2)] = 0 \\ \Rightarrow\; & Q_1 - Q_3 = Q_4 - Q_2 = 0 \\ \Rightarrow\; & Q_1 = Q_3 \\ & Q_2 = Q_4. \end{aligned}$$

If we assume $P_0 = 0$ as the base pressure, equation $Q_1 = Q_3$ or $Q_2 = Q_4$ produces

$$P_S = P_1 + P_2;$$

and combining with

$$P_L = P_1 - P_2;$$

we can solve for

$$\begin{aligned} P_1 &= \frac{P_S + P_L}{2} \\ P_2 &= \frac{P_S - P_L}{2}. \end{aligned}$$

The supply flow $Q_S$ is

$$Q_S = Q_1 + Q_2 = Q_3 + Q_4.$$

To summarize, we can get the two flows $Q_L$ and $Q_S$ as:

$$
\begin{aligned}
Q_L &= C_d A_1 \sqrt{\frac{1}{\rho}(P_S - P_L)} - C_d A_2 \sqrt{\frac{1}{\rho}(P_S + P_L)}\,; \\
Q_S &= C_d A_1 \sqrt{\frac{1}{\rho}(P_S - P_L)} + C_d A_2 \sqrt{\frac{1}{\rho}(P_S + P_L)}\,:
\end{aligned}
$$

Since the orifice areas $A_i$ are functions of $x_v$, we can get the supply and load flows as functions of load pressure and valve position,

$$
\begin{aligned}
Q_S &= Q_S(x_v; P_L)\,; \\
Q_L &= Q_L(x_v; P_L)\,;
\end{aligned}
$$

expressions that are quite nonlinear.

## 3.3 Valve Coefficients

We wish to linearize $Q_L = Q_L(x_v; P_L)$ about a particular operating point 1. Using Taylor series expansion we get:

$$
\Delta Q_L = \left(\frac{\partial Q_L}{\partial x_v}\right)_1 \Delta x_v + \left(\frac{\partial Q_L}{\partial P_L}\right)_1 \Delta P_L\,:
$$

We define:

$$
\begin{aligned}
&\text{flow gain} \qquad K_q = \frac{\partial Q_L}{\partial x_v} \\
&\text{flow–pressure coefficient} \qquad K_c = -\frac{\partial Q_L}{\partial P_L} \\
&\text{pressure sensitivity} \qquad K_p = \frac{\partial P_L}{\partial x_v} = \frac{K_q}{K_c}\,:
\end{aligned}
$$

Therefore, the linearized equation of pressure–flow curves becomes:

$$
\Delta Q_L = K_q \Delta x_v - K_c \Delta P_L\,:
$$

With regards to the above flow coefficients:

- $K_q$ affects open–loop system gain and is max at the zero operating point,
- $K_c$ affects system damping ratio and is min at the zero operating point.

With regards to the operating curves $Q_L = Q_L(x_v; P_L)$:

- $P_L$ is set by load demand,
- $Q_L$ is set by valve stroke at that load.

## 3.4 Critical Center Valves

Ideally, leakage flows are zero for critical center valves,

$$
\begin{aligned}
Q_2 &\ll Q_1 \\
Q_4 &\ll Q_3
\end{aligned}
$$

for $x_v > 0$. Therefore,

$$
Q_L = Q_1 = C_d A_1 \sqrt{\frac{2}{\rho}\left(\frac{P_S - P_L}{2}\right)} .
$$

For a valve stroke the other way, $x_v < 0$, we have

$$
\begin{aligned}
Q_1 &\ll Q_2 \\
Q_3 &\ll Q_4
\end{aligned}
$$

and

$$
Q_L = -Q_4 = -C_d A_2 \sqrt{\frac{2}{\rho}\left(\frac{P_S + P_L}{2}\right)} .
$$

Therefore, in general for symmetrical orifices,

$$
Q_L = C_d |A_1| \frac{x_v}{|x_v|} \sqrt{\frac{1}{\rho}\left(P_S - \frac{x_v}{|x_v|} P_L\right)} .
$$

The valve area, $A_1$, is in general function of $x_v$,

$$
A_1 = A_1(x_v) ;
$$

or

$$
dA_1 = \left(\frac{dA_1}{dx_v}\right) dx_v ;
$$

where

$$
\frac{dA_1}{dx_v} \equiv w
$$

and is called the valve area gradient. Integrating the last equation,

$$
A_1 - A_1(0) = \int_0^{x_v} w \, dx_v .
$$

For a critical center valve,

$$
A_1(0) = 0
$$

and the valve area gradient $w$ is constant. Therefore,

$$
A_1 = w x_v .
$$

For example, for a circular valve with diameter $d$ and full periphery ports,

$$
A_v = \pi d x_v \Rightarrow w = \pi d ;
$$

and

$$Q_L = C_d w |x_v| \frac{x_v}{|x_v|} \sqrt{\frac{1}{\rho}\left(P_S - \frac{x_v}{|x_v|} P_L\right)} \, .$$

This is the desired equation for the critical center valve operating curves, Figure 19.

The flow gain is the "slope" (spacing) of the curves with respect to $x_v$ for constant $P_L$,

$$K_q = \frac{\partial Q_L}{\partial x_v} = C_d w \sqrt{\frac{1}{\rho}(P_S - P_L)}$$

and this is constant. The flow–pressure coefficient is the slope of the curves with respect to $P_L$ at a fixed $x_v$,

$$K_c = -\frac{\partial Q_L}{\partial P_L} = \frac{C_d w x_v \sqrt{(1/\rho)(P_S - P_L)}}{2(P_S - P_L)}$$

and this depends on $P_L$. The pressure sensitivity coefficient is

$$K_p = \frac{K_q}{K_c} = \frac{2(P_S - P_L)}{x_v} \, .$$

At the null point,

$$Q_L = P_L = x_v = 0$$

and

$$\begin{aligned} K_{q0} &= C_d w \sqrt{\frac{P_S}{\rho}} \\ K_{c0} &= 0 \, . \end{aligned}$$

Computations for flow gain $K_q$ are very reliable, therefore stability characteristics of hydraulic systems are quite robust. The computed values for the flow–pressure coefficient may be far from reality; the main reason for this non–zero leakage flow. Leakage flow is maximum at valve neutral (null point) and decreases rapidly with $x_v$ as the spool lands overlap the valve orifices.

## 3.5 Open Center Valves

Consider the open center valve shown in Figure 20 and suppose that valve operation remains in the underlap region. We also assume matched,

$$A_1 = A_3 \, ; \quad A_2 = A_4$$

and symmetrical,

$$A_1(x_v) = A_2(-x_v) \, ;$$

valves. Then for underlap operation,

$$|x_v| \leq U$$

we have

$$A_1 = A_3 = w(U + x_v)$$
$$A_2 = A_4 = w(U - x_v)$$

where $U$ is the max underlap amount.

The flow through the valve is given by

$$Q_L = C_d A_1 \sqrt{\frac{1}{\rho}(P_S - P_L)} - C_d A_2 \sqrt{\frac{1}{\rho}(P_S + P_L)};$$

and normalized,

$$\frac{Q_L}{C_d w U \sqrt{P_S/\rho}} = \left(1 + \frac{x_v}{U}\right)\sqrt{1 - \frac{P_L}{P_S}} - \left(1 - \frac{x_v}{U}\right)\sqrt{1 + \frac{P_L}{P_S}};$$

which is the expression for the desired flow–pressure curves. Outside the underlap region, the open center valve behaves like a critical center. The maximum flow through the valve is

$$Q_{L_{max}} = Q_L(x_{v_{max}}, 0) = 2C_d w U \sqrt{P_S/\rho} \quad \text{at} \quad x_{v_{max}} = U.$$

Note that for a critical center valve at $x_{v_{max}} = U$ we would have,

$$Q_{L_{max}} = C_d w U \sqrt{P_S/\rho};$$

half as much for the open center.

The flow gain is

$$K_q = \frac{\partial Q_L}{\partial x_v} = C_d w \sqrt{\frac{P_S}{\rho}} \left( \sqrt{1 - \frac{P_L}{P_S}} + \sqrt{1 + \frac{P_L}{P_S}} \right);$$

and at the null point,

$$K_{q0} = 2C_d w \sqrt{\frac{P_S}{\rho}};$$

twice that of the corresponding critical center valve. This means that the slope of the initial segment of the $(Q_L, x_v)$ curve of Figure 17 is twice as much as the subsequent segment. The flow pressure coefficient is,

$$K_c = -\frac{\partial Q_L}{\partial P_L} = C_d w \sqrt{\frac{P_S}{\rho}} \left(\frac{U}{2P_S}\right) \left( \frac{1 + \frac{x_v}{U}}{\sqrt{1 - \frac{P_L}{P_S}}} + \frac{1 - \frac{x_v}{U}}{\sqrt{1 + \frac{P_L}{P_S}}} \right).$$

At the null point,

$$K_{c0} = c_d w \sqrt{\frac{P_S}{\rho}} \left(\frac{U}{P_S}\right) \neq 0 \quad \text{as in the critical center valve.}$$

The pressure sensitivity coefficient is

$$K_p = \frac{K_q}{K_c}$$

and at the null point

$$K_{p0} = \frac{2P_S}{U} \neq 0 \quad \text{as in the critical center valve.}$$

Note that by taking the $\lim_{U \to 0}$; i.e., as the open center valve approaches critical center, we get the right result for $K_{c0}$ and $K_{p0}$ but not for $K_{q0}$. For the latter case, $U = 0$ has to be substituted before forming the relation for $Q_L$.

## 3.6 Flapper Valves

As we have already mentioned, the primary advantage of flapper valves is their loose tolerance requirements which lowers their cost. Due to their increased leakage, however, their use is restricted to low power applications.

Consider the single jet flapper shown in Figure 21. Continuity gives,

$$Q_1 = Q_2 + Q_L.$$

The orifice flows are

$$Q_1 = A_0 C_{d0} \sqrt{\frac{2}{\rho}(P_S - P_C)};$$

or

$$Q_1 = \frac{\pi}{4} D_0^2 C_{d0} \sqrt{\frac{2}{\rho}(P_S - P_C)};$$

and

$$Q_2 = A_f C_{df} \sqrt{\frac{2}{\rho} P_C} = \pi D_N (x_{f0} - x_f) C_{df} \sqrt{\frac{2}{\rho} P_C}.$$

If the load is blocked,

$$Q_1 = Q_2$$

we get

$$\frac{P_C}{P_S} = \left[1 + \left(\frac{C_{df} A_f}{C_{d0} A_0}\right)^2\right]^{-1}.$$

A design criterion is an equilibrium control pressure

$$P_C = 0.5 P_S.$$

Therefore, at equilibrium; i.e., the null point,

$$C_{df} A_f = C_{d0} A_0 \Rightarrow C_{d0} A_0 = C_{df} \pi D_N x_{f0}.$$

Substituting the expressions for $Q_1$ and $Q_2$ into

$$Q_L = Q_1 - Q_2$$

we get

$$\frac{Q_L}{C_{d0}A_0\sqrt{(2/\rho)P_S}} = \sqrt{1 - \frac{P_C}{P_S}} - \frac{C_{df}\pi D_N x_{f0}}{C_{d0}A_0}\left(1 - \frac{x_f}{x_{f0}}\right)\sqrt{\frac{P_C}{P_S}};$$

and for the design criterion

$$P_C = 0.5 P_S$$

we get

$$\frac{Q_L}{C_{d0}A_0\sqrt{(2/\rho)P_S}} = \sqrt{1 - \frac{P_C}{P_S}} - \left(1 - \frac{x_f}{x_{f0}}\right)\sqrt{\frac{P_C}{P_S}}:$$

The null coefficients, evaluated at

$$x_f = Q_L = 0; \quad P_C = 0.5 P_S$$

are

$$K_{q0} = \left.\frac{\partial Q_L}{\partial x_f}\right|_0 = C_{df}\pi D_N\sqrt{\frac{P_S}{\rho}};$$

$$K_{c0} = -\left.\frac{\partial Q_L}{\partial P_c}\right|_0 = \frac{2C_{df}\pi D_N x_{f0}}{\sqrt{\rho P_S}};$$

$$K_{p0} = \left.\frac{\partial P_c}{\partial x_f}\right|_0 = \frac{P_S}{2x_{f0}}:$$

For the double jet flapper valve shown in Figure 22 we have,

$$Q_L = Q_1 - Q_2;$$
$$Q_L = Q_4 - Q_3;$$

and substituting in for the flows,

$$Q_L = C_{d0}A_0\sqrt{\frac{2}{\rho}(P_S - P_1)} - C_{df}\pi D_N(x_{f0} - x_f)\sqrt{\frac{2}{\rho}P_1};$$

$$Q_L = C_{df}\pi D_N(x_{f0} + x_f)\sqrt{\frac{2}{\rho}P_2} - C_{d0}A_0\sqrt{\frac{2}{\rho}(P_S - P_2)}:$$

At the null point,

$$C_{d0}A_0 = C_{df}\pi D_N x_{f0}$$

they become

$$\frac{Q_L}{C_{d0}A_0\sqrt{P_S/\rho}} = \sqrt{2\left(1 - \frac{P_1}{P_S}\right)} - \left(1 - \frac{x_f}{x_{f0}}\right)\sqrt{\frac{2P_1}{P_S}};$$

$$\frac{Q_L}{C_{d0}A_0\sqrt{P_S/\rho}} = \left(1 + \frac{x_f}{x_{f0}}\right)\sqrt{\frac{2P_2}{P_S}} - \sqrt{2\left(1 - \frac{P_2}{P_S}\right)};$$

and combined with

$$P_L = P_1 - P_2$$

then define, implicitly, the operating curves $Q_L(x_f; P_L)$.

In order to evaluate the valve coefficients at the null point,

$$x_f = Q_L = P_L = 0\,; \quad P_1 = P_2 = 0.5P_S\,;$$

we can linearize all three equations

$$\begin{aligned}
\Delta Q_L &= C_{df}\pi D_N \sqrt{\frac{P_S}{\rho}}\,\Delta x_f - \frac{2C_{df}\pi D_N x_{f0}}{\sqrt{\rho P_S}}\,\Delta P_1\,;\\
\Delta Q_L &= C_{df}\pi D_N \sqrt{\frac{P_S}{\rho}}\,\Delta x_f + \frac{2C_{df}\pi D_N x_{f0}}{\sqrt{\rho P_S}}\,\Delta P_2\,;\\
\Delta P_L &= \Delta P_1 - \Delta P_2\,.
\end{aligned}$$

Therefore,

$$\Delta Q_L = C_{df}\pi D_N \sqrt{\frac{P_S}{\rho}}\,\Delta x_f - \frac{C_{df}\pi D_N x_{f0}}{\sqrt{\rho P_S}}\,\Delta P_L\,;$$

or

$$\begin{aligned}
K_{q0} &= C_{df}\pi D_N \sqrt{\frac{P_S}{\rho}} = \text{ same as single jet}\,;\\
K_{c0} &= \frac{C_{df}\pi D_N x_{f0}}{\sqrt{\rho P_S}} = \text{ half the single jet}\,;\\
K_{p0} &= \frac{P_S}{x_{f0}} = \text{ twice the single jet}\,.
\end{aligned}$$

## 3.7 Valve Flow Forces

1. Momentum balance: Valve flow forces arise because of two main reasons:

- the acceleration of the fluid as it passes through the valve chambers, with the valve spool held stationary, and
- the acceleration of the fluid within the valve chamber when the flow rate is changed.

Consider the valve cross section shown in Figure 23. Application of the momentum theorem gives,

$$\sum \vec{F} = \frac{\partial}{\partial t}\int_V \rho\vec{v}\,dv + \iint_A \vec{v}\,(\rho\vec{v}\cdot\vec{n})\,dA\,.$$

We define the acceleration length by

$$L = \frac{\int_V \rho\vec{v}\,dv}{\rho Q}\,.$$

Physically, $L$ represents the length of the fluid that is accelerated when the flow rate $Q$ is changed and is of the order of the distance between the inlet and outlet ports of the chamber. Considering the x–component of the total force, the momentum equation is written as,

$$\sum F_x = \rho L \frac{\partial Q}{\partial t} + \rho U_2^2 A_2 \cos\theta .$$

The first term on the right–hand side leads to "transient flow forces", and the second to "steady–state flow forces".

2. <u>Transient flow forces:</u> For the orifice flow,

$$Q = C_d w x_v \sqrt{\frac{2}{\rho}(P_1 - P_2)} .$$

Therefore,

$$\frac{\partial Q}{\partial t} = C_d w \sqrt{\frac{2}{\rho}} \left[ \sqrt{P_1 - P_2}\, \frac{\partial x_v}{\partial t} + \frac{x_v}{2\sqrt{P_1 - P_2}} \frac{\partial (P_1 - P_2)}{\partial t} \right] .$$

The transient flow force is

$$\rho L \frac{\partial Q}{\partial t} .$$

Since the flow may be changed by varying either $x_v$ or $P_1 - P_2$, the transient flow force involves the rate of change of both of these terms.

3. <u>Steady state flow force:</u> Recall that the discharge coefficient $C_d$ is given by the product of the velocity coefficient, $C_v$, and contraction coefficient, $C_c$,

$$C_d = C_v C_c ;$$

and

$$U_2 = \frac{Q}{A_2} ;$$
$$A_2 = C_c w x_v .$$

The steady–state flow force is then given by:

$$\rho U_2^2 A_2 \cos\theta = \rho \frac{Q^2}{A_2} \cos\theta = 2 C_d C_v w x_v (P_1 - P_2) \cos\theta .$$

Typical values of angle $\theta$ vs. $x_v$ curves are shown in Figure 24.

4. <u>Total flow force:</u> Combining the previous equations, we get

$$\sum F_x = L C_d w \sqrt{2\rho} \left[ \sqrt{P_1 - P_2}\, \frac{\partial x_v}{\partial t} + \frac{x_v}{2\sqrt{P_1 - P_2}} \frac{\partial (P_1 - P_2)}{\partial t} \right] + 2 C_d C_v w x_v (P_1 - P_2) \cos\theta .$$

Steady–state forces are always stabilizing forces: the change in force accompanying a change in valve stroke tends to resist that change. Transient forces may be either stabilizing or destabilizing.

5. <u>Force balance for a critical center spool valve:</u> Consider a positive (downward, as in Figure 18) stroke of the valve. In the upper (high pressure) chamber the fluid accelerates upward and the reaction force is downward in the same direction as the stroke. This is destabilizing whereas in the lower (low pressure) chamber the directions are opposite and the transient force is stabilizing. The operating pressure drops in the upper and lower chambers are $P_S - P_1$ and $P_2 - P_0 = P_2$, respectively, so that repeated application of the above equations gives the following forces due to the flow:

- transient, upper chamber:

$$-L_1 C_d w \sqrt{2\rho}\left[\sqrt{P_S - P_1}\,\frac{\partial x_v}{\partial t} + \frac{x_v}{2\sqrt{P_S - P_1}}\cdot\frac{\partial (P_S - P_1)}{\partial t}\right]$$

- transient, lower chamber:

$$L_2 C_d w \sqrt{2\rho}\left(\sqrt{P_2}\,\frac{\partial x_v}{\partial t} + \frac{x_v}{2\sqrt{P_2}}\cdot\frac{\partial P_2}{\partial t}\right)$$

- steady state, upper chamber:

$$2 w x_v C_d C_v (P_S - P_1)\cos\theta$$

- steady state, lower chamber:

$$2 w x_v C_d C_v P_2 \cos\theta$$

where an upward reaction force, opposite to the stroking force, is taken as positive.

The pressure drops are

$$P_1 = \frac{1}{2}(P_S + P_L);$$
$$P_2 = \frac{1}{2}(P_S - P_L).$$

Then, the total force for this valve is:

$$F_f = 2 w x_v C_d C_v (P_S - P_L)\cos\theta + \rho(L_2 - L_1) Q_L \left[\frac{1}{x_v}\cdot\frac{\partial x_v}{\partial t} - \frac{1}{2(P_S - P_L)}\cdot\frac{\partial P_L}{\partial t}\right]$$

where

$$Q_L = C_d w x_v \sqrt{\frac{P_S - P_L}{\rho}}.$$

The first part is the steady state force, and the second part is the transient force.

Adding the inertia of the spool mass, $M_s$, the stroking force may be written as

$$F_i = M_s \frac{\partial^2 x_v}{\partial t^2} + B_f \frac{\partial x_v}{\partial t} + K_f x_v - B_p \frac{\partial P_L}{\partial t},$$

where:

$$B_f = \frac{\rho (L_2 - L_1) Q_L}{x_v} = C_d w (L_2 - L_1) \sqrt{\rho (P_S - P_L)}$$

is the damping coefficient due to transient flow force;

$$K_f = 2 w C_d C_v (P_S - P_L) \cos\theta \approx 0.43 w (P_S - P_L)$$

is the spring constant due to steady–state flow force;

$$B_p = \frac{\rho (L_2 - L_1) Q_L}{2 (P_S - P_L)} = \frac{C_d w x_v (L_2 - L_1)}{2} \sqrt{\frac{\rho}{P_S - P_L}}$$

is the load feedback due to transient flow force. Valve dynamics are generally analyzed with $P_L$ invariant and only the valve stroke $x_v$ as an input. In that case the last term is normally neglected. Attempt is also made to make $L_2 \approx L_1$ in order to eliminate the small and somewhat unpredictable effects associated with the transient flow forces.

6. <u>Flapper valve flow forces:</u> In order to apply the momentum theorem, consider the control volume surrounding the interaction region and shown with dashed lines in Figure 25. $F_f$ is an external force which is applied to hold the flapper in position. Transient flow forces are assumed to be negligible so that the momentum theorem for this problem is

$$\sum \vec{F} = \iint_A \vec{V} (\rho \vec{V} \cdot \hat{n})\, dA \,;$$

where $\hat{n}$ is the outward–pointing unit normal vector at each point on the surface $A$ of the control volume. For the upward direction

$$\sum F = (P_a - P) A_N + F_f$$

and

$$\iint_A V (\rho \vec{V} \cdot \hat{n})\, dA = -u [\rho u (-1)] A_N = \rho u^2 A_N \,.$$

Combining these results with

$$A_N = \frac{\pi D^2}{4} \,;$$

we get

$$F_f = (P - P_a) A_N + \rho u^2 A_N \,.$$

If losses in the jet supply duct are neglected, Bernoulli's equation may be used to give

$$P = P_c - \frac{1}{2} \rho u^2$$

and if the pressures are expressed as gage values $P_a = 0$. The result is

$$F_f = \left( P_c + \tfrac{1}{2} \rho u^2 \right) A_N \,.$$

The jet exit velocity $u$ may be expressed in terms of the flapper geometry by means of the discharge coefficient:

$$u = \frac{Q}{A_N} = C_{df} \frac{A_f}{A_N} \sqrt{\frac{2 P_c}{\rho}} \,;$$

and

$$\frac{1}{2}\rho u^2 A_N = C_{df}^2 \frac{A_f^2}{A_N} P_c = 4\pi C_{df}^2 h^2 P_c .$$

Therefore,

$$F_f = P_c A_N + 4\pi C_{df}^2 h^2 P_c = P_c A_N \left(1 + \frac{16 C_{df}^2 h^2}{D_N^2}\right) .$$

It is important to remember that this expression is for the upward force necessary to hold the flapper in place against the downward flow from the upper jet.

7. Application to the double jet flapper: For the double jet flapper shown in Figure 26 there are two relevant flow forces, one from each jet.

The downward force on the flapper due to the upper jet is found by substituting

$$P_c = P_1 ; \quad \text{and} \quad h = x_{f0} - x_f .$$

We get then,

$$F_1 = P_1 A_N + 4\pi C_{df}^2 (x_{f0} - x_f)^2 P_1 .$$

Similarly, for the upward force due to the lower jet

$$F_2 = P_2 A_N + 4\pi C_{df}^2 (x_{f0} + x_f)^2 P_2 .$$

The net downward force is therefore,

$$F_1 - F_2 = (P_1 - P_2) A_N + 4\pi C_{df}^2 \left[ (x_{f0} - x_f)^2 P_1 - (x_{f0} + x_f)^2 P_2 \right] .$$

Note that if

$$P_1 = P_2$$

then

$$F_1 - F_2 = -16\pi C_{df} P_1 x_{f0} x_f ;$$

indicating that for positive flapper deflection the dynamic effect of the flow deflection is to create a net upward force (destabilizing) on the flapper. In other words, the flapper is "attracted" to the throttled jet under the influence of what can be thought of as a negative spring constant.

If the expression in the brackets above is expanded, using

$$P_L = P_1 - P_2$$

we get:

$$\begin{aligned} F_1 - F_2 &= P_L A_N + 4\pi C_{df}^2 x_{f0}^2 \left\{ \left[ 1 + \left(\frac{x_f}{x_{f0}}\right)^2 \right] P_L - 2 \frac{x_f}{x_{f0}} (P_1 - P_2) \right\} \\ &= P_L A_N \left\{ 1 + 4\pi C_{df}^2 \frac{x_{f0}^2}{A_N} \left[ 1 + \left(\frac{x_f}{x_{f0}}\right)^2 \right] - 8\pi C_{df}^2 \frac{x_{f0} x_f}{A_N} \cdot \frac{P_1 + P_2}{P_L} \right\} . \end{aligned}$$

By design,

$$\frac{x_{f0}^2}{A_N} \ll 1$$

so that the second term in the curly brackets may be neglected. If this expression is evaluated near center conditions

$$(P_1 + P_2 \approx P_S)$$

the final result for design purposes is

$$F_1 - F_2 = P_L A_N - 8\pi C_{df}^2 P_S x_{f0} x_f \, .$$

Note again that $F_1 - F_2$ is the net downward force; i.e., in the direction opposite to positive flapper motion.

# 4 HYDRAULIC POWER ELEMENTS

## 4.1 Introduction

So far, we have seen the following:

1. Fluids/Flows:
   - Effective bulk modulus.
   - Orifice flows.
   - Leakage flows.
   - For laminar flow: $Q \sim \Delta P$.
   - For turbulent flow: $Q \sim \sqrt{\Delta P}$.
2. Actuators:
   - Ideal.
   - Losses: Torque — Flow.
   - Efficiencies.
3. Valves:
   - Spool, critical center, open center.
   - Flapper.
   - Valve operating curves: $Q_L = Q_L(x_v; P_L)$.
   - Valve coefficients: $\Delta Q_L = K_q \Delta x_v - K_c \Delta P_L$.

Combination of valve, actuator, and load characteristics produces the so–called hydraulic power element, schematically shown in Figure 27. In general, there are four types of hydraulic power elements:

- Valve controlled motor, VCM.
- Valve controlled piston, VCP.
- Pump controlled motor, PCM.
- Pump controlled piston, PCP.

We present the analysis of the VCM in detail in the following section.

## 4.2 Valve Controlled Motor

Consider the VCM shown in Figures 28 and 29. We utilize the continuity equation

$$\sum Q_{in} - \sum Q_{out} = \frac{dV}{dt} + \frac{V}{\beta} \cdot \frac{dP}{dt};$$

where

$$\begin{aligned} V &= \text{volume} \\ P &= \text{pressure} \\ \beta &= \text{effective bulk modulus} \end{aligned}$$

and unlike our previous applications of continuity, here we include compressibility effects. Applying this equation to Figure 28 we get,

$$\begin{aligned} Q_1 - C_{im}(P_1 - P_2) - C_{em} P_1 &= \frac{dV_1}{dt} + \frac{V_1}{\beta_e} \cdot \frac{dP_1}{dt}; \\ C_{im}(P_1 - P_2) - C_{em} P_2 - Q_2 &= \frac{dV_2}{dt} + \frac{V_2}{\beta_e} \cdot \frac{dP_2}{dt}. \end{aligned}$$

We need to express the volumes $V_1$ and $V_2$ in terms of motor parameters.

With reference to Figure 30 we have,

$$V_1 + V_2 = 2V_0 = \text{const.}$$

Differentiating,

$$\frac{dV_1}{dt} = -\frac{dV_2}{dt} = D_m \dot{\theta}_m;$$

and integrating,

$$\begin{aligned} V_1 &= V_0 + f(\theta_m); \\ V_2 &= V_0 - f(\theta_m). \end{aligned}$$

The load flow is

$$Q_L = \frac{Q_1 + Q_2}{2};$$

and substituting in the values of $Q_1$ and $Q_2$,

$$\begin{aligned} Q_L &= D_m \dot{\theta}_m + \left( C_{im} + \frac{C_{em}}{2} \right) (P_1 - P_2) + \frac{V_0}{2\beta_e} \cdot \frac{d(P_1 - P_2)}{dt} \\ &\quad + \frac{f(\theta_m)}{2\beta_e} \left( \frac{dP_1}{dt} + \frac{dP_2}{dt} \right). \end{aligned}$$

Since

$$P_1 + P_2 = P_S = \text{const.}$$

we have

$$\frac{dP_1}{dt} + \frac{dP_2}{dt} = 0$$

and if we denote

$$V_t \equiv 2V_0$$

we get

$$Q_L = D_m \dot{\theta}_m + C_{tm} P_L + \frac{V_t}{4\beta_e} \cdot \frac{dP_L}{dt};$$

with

$$C_{tm} = C_{im} + \frac{C_{em}}{2}.$$

For the valve we have,

$$Q_L = K_q x_v - K_c P_L.$$

We need a load description such that $P_L$ is determined by torque requirements. The torque delivered to the load is $P_L D_m$. Therefore, torque balance gives,

$$P_L D_m = J_t \ddot{\theta}_m + B_m \dot{\theta}_m + G\theta_m + T_L;$$

where $T_L$ is a static load torque. In summary, the equations in the s-domain are

$$\begin{aligned} Q_L &= D_m \theta_m s + C_{tm} P_L + \frac{V_t}{4\beta_e} P_L s; \\ Q_L &= K_q x_v - K_c P_L; \\ P_L D_m &= J_t \theta_m s^2 + B_m \theta_m s + G\theta_m + T_L; \end{aligned}$$

or

$$\begin{aligned} D_m \theta_m s + K_{ce} P_L + C_{cm} P_L s &= K_q x_v; \\ P_L D_m &= J_t \theta_m s^2 + B_m \theta_m s + G\theta_m + T_L; \end{aligned}$$

where

$$\begin{aligned} K_{ce} &= K_c + C_{tm}; \\ C_{cm} &= \frac{V_t}{4\beta_e}. \end{aligned}$$

Symbolically, we can see that we get an expression of the form,

$$\theta_m = f(x_v; T_L);$$

where $\theta_m$ is the output of the system and $x_v$, $T_L$ the two inputs. Schematically, this block diagram is shown in Figure 31. An expanded block diagram is shown in Figure 32, where the three basic elements, valve, motor, and load can be identified. Since there are two inputs to the system, we can evaluate two distinct transfer functions:

- Valve position input, $x_v$, $T_L = 0$:

$$\frac{\theta_m}{x_v} = \frac{K_q/D_m}{s + \frac{1}{D_m^2}(K_{ce} + C_{cm}s)(J_t s^2 + B_m s + G)}:$$

- Load torque input, $T_L$, $x_v = 0$:

$$\frac{\theta_m}{T_L} = \frac{-\frac{1}{D_m^2}(K_{ce} + C_{cm}s)}{s + \frac{1}{D_m^2}(K_{ce} + C_{cm}s)(J_t s^2 + B_m s + G)}:$$

<u>Simplifications to characteristic equation:</u>
The general form of the characteristic equation is,

$$s + \frac{1}{D_m^2}(K_{ce} + C_{cm}s)(J_t s^2 + B_m s + G) = 0:$$

Spring loads are usually negligible,

$$G = 0;$$

so

$$s\left[1 + \frac{K_{ce}B_m}{D_m^2}\left(1 + \frac{C_{cm}}{K_{ce}}s\right)\left(1 + \frac{J_t}{B_m}s\right)\right] = 0:$$

Usually,

$$\frac{K_{ce}B_m}{D_m^2} \ll 1;$$

so that the characteristic equation becomes

$$s\left\{1 + \frac{K_{ce}B_m}{D_m^2}\left[\frac{C_{cm}J_t}{K_{ce}B_m}s^2 + \left(\frac{J_t}{B_m} + \frac{C_{cm}}{K_{ce}}\right)s\right]\right\} = 0:$$

This has the form,

$$s\left(\frac{s^2}{\omega_h^2} + 2\frac{\delta_h}{\omega_h}s + 1\right) = 0;$$

which represents a type–1 system where

$$\omega_h = \sqrt{\frac{D_m^2}{J_t C_{cm}}};$$

is the hydraulic undamped natural frequency, and

$$\delta_h = \frac{1}{2}\left(\frac{K_{ce}}{D_m}\sqrt{\frac{J_t}{C_{cm}}} + \frac{B_m}{D_m}\sqrt{\frac{C_{cm}}{J_t}}\right);$$

is the hydraulic damping ratio. Recall that,

$$C_{cm} = \frac{V_t}{4\beta_e};$$

and the significance of the effective bulk modulus on hydraulic natural frequency is evident. The hydraulic spring constant is

$$K_h = \frac{D_m^2}{C_{cm}} = \frac{4\beta_e D_m^2}{V_t};$$

and the hydraulic damping ratio, for $B_m = 0$,

$$\delta_h = \frac{1}{2} \cdot \frac{K_{ce}}{D_m}\sqrt{\frac{J_t}{C_{cm}}} = \frac{K_{ce}}{D_m}\sqrt{\frac{J_t\beta_e}{V_t}}.$$

To summarize, the transfer functions are written as:

- Due to valve position:

$$\frac{\theta_m}{x_v} = \frac{\frac{K_q}{D_m}}{s\left(\frac{s^2}{\omega_h^2} + 2\frac{\delta_h}{\omega_h}s + 1\right)}.$$

- Due to load torque:

$$\frac{\theta_m}{T_L} = \frac{-\frac{K_{ce}}{D_m^2}\left(1 + \frac{V_t}{4\beta_e K_{ce}}s\right)}{s\left(\frac{s^2}{\omega_h^2} + 2\frac{\delta_h}{\omega_h}s + 1\right)}.$$

At very low $\omega$ inputs we see that

$$\theta_m \approx \frac{K_q}{D_m}x_v;$$

so that the term $K_q/D_m$ is an expression of the steady state gain of the system. For critical center valves,

$$K_q = C_d W\sqrt{\frac{P_S}{\rho}}\sqrt{1 - \frac{P_L}{P_S}};$$

so that the steady state gain will decrease at loads away from null. At the maximum power setting

$$P_L = \frac{2}{3}P_S;$$

so that

$$\frac{K_q}{K_{q\,(\mathrm{max\ power})}} = 1.73;$$

which amounts to a 58% decrease in flow gain from null to maximum power. Thus a positive gain margin at null load conditions will ensure stability at higher loads. Conversely, a system that is stable under load will not necessarily be stable under no load conditions.

Consider the transfer function due to load torque: $\theta_m / T_L$. It is usually called “dynamic compliance” since $-T_L/\theta_m$ is a “stiffness”. For slowly varying inputs we see that

$$T_L \approx -\frac{D_m^2}{K_{ce}} \dot{\theta}_m \, .$$

Since the ratio $D_m^2 / K_{ce}$ is usually very large, a small decrease in motor speed leads to a large increase in the resisting load torque: this means that the hydraulic system is quite “stiff”.

## 4.3 VCM in State Space

The governing equations are:

- valve flow:

$$Q_L = K_q x_v - K_c P_L \, ;$$

- motor flow demand:

$$Q_L = D_m \dot{\theta}_m + C_{tm} P_L + C_{cm} \dot{P}_L \, ;$$

- load demand:

$$P_L D_m = J_t \ddot{\theta}_m + B_m \dot{\theta}_m + G \theta_m + T_L \, .$$

We eliminate $Q_L$ and rearrange,

$$\begin{aligned} \dot{P}_L &= \frac{1}{C_{cm}} (-D_m \dot{\theta}_m - K_{ce} P_L + K_q x_v) \, ; \\ \ddot{\theta}_m &= \frac{1}{J_t} (-B_m \dot{\theta}_m - G \theta_m + D_m P_L - T_L) \, . \end{aligned}$$

If we define,

$$\begin{aligned} \text{state vector} \quad x &= \begin{bmatrix} \dot{\theta}_m & \theta_m & P_L \end{bmatrix}^T \\ \text{input vector} \quad u &= \begin{bmatrix} x_v & T_L \end{bmatrix}^T \\ \text{output vector} \quad y &= \begin{bmatrix} Q_L & \dot{\theta}_m & \theta_m & P_L \end{bmatrix}^T \end{aligned}$$

the state space equations are:

$$\begin{bmatrix} \ddot{\theta}_m \\ \dot{\theta}_m \\ \dot{P}_L \end{bmatrix} = \begin{bmatrix} -B_m/J_t & -G/J_t & D_m/J_t \\ 1 & 0 & 0 \\ -D_m/C_{cm} & 0 & -K_{ce}/C_{cm} \end{bmatrix} \begin{bmatrix} \dot{\theta}_m \\ \theta_m \\ P_L \end{bmatrix} + \begin{bmatrix} 0 & -1/J_t \\ 0 & 0 \\ K_q/C_{cm} & 0 \end{bmatrix} \begin{bmatrix} x_v \\ T_L \end{bmatrix} \, ;$$

and

$$\begin{bmatrix} Q_L \\ \dot{\theta}_m \\ \theta_m \\ P_L \end{bmatrix} = \begin{bmatrix} 0 & 0 & -K_c \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \dot{\theta}_m \\ \theta_m \\ P_L \end{bmatrix} + \begin{bmatrix} K_q & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_v \\ T_L \end{bmatrix};$$

or, in compact notation,

$$\begin{aligned} \dot{x} &= Ax + Bu; \\ y &= Cx + Du: \end{aligned}$$

## 4.4 Valve Controlled Piston

Although it is possible to go through the analysis of the valve controlled piston (VCP), shown in Figure 33, in the same way as for the VCM, it is easier to write the equations directly by enforcing the analogies between rotational and translational systems:

$$\begin{aligned} J_t &\longleftrightarrow M_t \\ D_m &\longleftrightarrow A_p \quad \text{volume displaced per unit motion} \\ \theta_m &\longleftrightarrow x_p \\ G &\longleftrightarrow K \end{aligned}$$

The VCP equations then are,

$$\begin{aligned} Q_L &= K_q x_v - K_c P_L; \\ Q_L &= A_p s x_p + C_{tp} P_L + \frac{V_t}{4\beta_e} s P_L; \\ A_p P_L &= M_t s^2 x_p + B_p s x_p + K x_p + F_L: \end{aligned}$$

Both systems (VCM and VCP) have the same internal and external leakage characteristics and both systems are controlled by an idealized critical center valve. With moderate load damping ($B_p K_{ce}/A_p^2 \ll 1$) and no spring load we have:

$$\begin{aligned} &\text{Valve gain constant} \quad \frac{K_q}{A_p} \\ &\text{Natural frequency} \quad \omega_h^2 = \frac{4\beta_e A_p^2}{V_t M_t} \\ &\text{Fluid spring constant} \quad K_h = \frac{4\beta_e A_p^2}{V_t} \\ &\text{Damping ratio} \quad \delta_h = \frac{K_{ce}}{A_p}\sqrt{\frac{\beta_e M_t}{V_t}} + \frac{B_p}{4A_p}\left(\sqrt{\frac{V_t}{\beta_e M_t}}\right) \end{aligned}$$

The transfer function is,

$$x_p = \frac{(K_q/A_p)x_v - (K_{ce}/A_p^2)\,[1 + (s/2\delta_h \omega_h)]F_L}{s\,[(s/\omega_h)^2 + (2\delta_h/\omega_h)s + 1]}:$$

All previous remarks concerning the response and compliance of the VCM are applicable to the VCP with the application of these analogies.

## 4.5 Pump Controlled Motor

Pump controlled motors are used in applications requiring high horsepower. Compared to VCM's, however, they experience slower response. In the PCM the valve is replaced by a variable displacement pump and a replenishment system, as shown in Figure 34. The pump supplies high pressure fluid in response to load demands; the motor speed and direction of rotation may be controlled by varying the pump stroke. The replenishment system maintains a constant low return pressure, $P_r$. From a controls point of view the basic difference between valve control and pump control is that in the pump system only the high pressure side is changed to respond to changing loads.

The pump must supply its own and motor leakage flows, compressibility flows, and power flows, as shown in Figure 35. Applying continuity to the flows shown in the figure, we get

$$Q_p = D_m \dot{\theta}_m + (C_{ep} + C_{em})P_1 + (C_{ip} + C_{im})(P_1 - P_r) + \frac{V_0}{\beta_e} \cdot \frac{dP_1}{dt}.$$

If we define,

$$\begin{aligned} D_p &= \frac{Q_p}{N_p}; \\ K_p &= \frac{D_p}{\phi}; \\ C_t &= C_{ep} + C_{em} + C_{ip} + C_{im}; \\ C_{it} &= C_{ip} + C_{im}; \end{aligned}$$

where

$$\begin{aligned} D_p &= \text{volumetric displacement} \\ K_p &= \text{displacement gradient of pump} \\ N_p &= \text{pump speed} \\ \phi &= \text{pump stroke angle} \end{aligned}$$

we get

$$Q_p = K_p N_p \phi = D_m \dot{\theta}_m + C_t P_1 - C_{it} P_r + \frac{V_0}{\beta_e} \cdot \frac{dP_1}{dt}.$$

If $T_g$ is the torque generated by the motor, the torque balance equation becomes:

$$\begin{aligned} T_g &= \eta_t (P_1 - P_r) D_m; \\ T_g &= J_t \ddot{\theta}_m + B_m \dot{\theta}_m + G\theta_m + T_L; \end{aligned}$$

where an internal friction force $\mathrm{sign}(\dot{\theta}_m)\cdot(P_1+P_r)C_f D_m$ has been neglected. Therefore, if $\eta_t = 1$,

$$P_1 D_m = J_t\ddot{\theta}_m + B_m\dot{\theta}_m + G\theta_m + P_r D_m + T_L.$$

To summarize, the equations in the s-domain are

$$K_p N_p \phi + C_{it}P_r = D_m s\theta_m + C_t P_1 + \frac{V_0}{\beta_e} s P_1,$$

$$P_1 D_m = J_t s^2\theta_m + B_m s\theta_m + G\theta_m + P_r D_m + T_L.$$

If we assume $G = 0$ and $B_m C_t / D_m^2 \ll 1$, the response is

$$\theta_m = \frac{\frac{K_p N_p}{D_m}\phi - \frac{C_t}{D_m^2}\left(1+\frac{V_0}{\beta_e C_t}s\right)T_L - \frac{C_{et}}{D_m^2}\left(1+\frac{V_0}{\beta_e C_{et}}s\right)P_r}{s\left(\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1\right)},$$

where

$$\omega_h = \sqrt{\frac{\beta_e D_m^2}{V_0 J_t}},$$

$$\delta_h = \frac{1}{2}\cdot\frac{C_t}{D_m}\sqrt{\frac{\beta_e J_t}{V_0}} + \frac{1}{2}\cdot\frac{B_m}{D_m}\left(\sqrt{\frac{V_0}{\beta_e J_t}}\right).$$

Usually $P_r$ = const. and so we can assume it to be zero in the above transfer function, since in reality it represents $\Delta P_r$; i.e., deviations from a nominal value.

Note that if the valve and pump systems are of corresponding sizes,

$$V_t \approx 2V_0,$$

so that

$$\frac{\omega_{h(PCM)}}{\omega_{h(VCM)}} = \frac{\sqrt{\frac{\beta_e D_m^2}{V_0 J_t}}}{\sqrt{\frac{4\beta_e D_m^2}{V_t J_t}}} = \frac{1}{2}\sqrt{\frac{V_t}{V_0}} \approx \frac{\sqrt{2}}{2} = 0.707.$$

This is because the fluid spring in the high pressure side of the PCM is not balanced by a spring on the low pressure side as in the case of the VCM. The PCM is thus slower to respond than the VCM. Actually, $V_t < 2V_0$ because valves are smaller than pumps, and this aggravates the situation. However, since $K_p N_p$ is much more constant and predictable than $K_q$, PCM systems are more predictable with the above expressions. Note also that, if $B_m$ is negligible,

$$\frac{\delta_{h(PCM)}}{\delta_{h(VCM)}} = \frac{\frac{1}{2}\frac{C_t}{D_m}\sqrt{\frac{\beta_e J_t}{V_0}}}{\frac{K_{ce}}{D_m}\sqrt{\frac{\beta_e J_t}{V_t}}} = \frac{1}{2}\cdot\frac{C_t}{K_{ce}}\sqrt{\frac{V_t}{V_0}} \approx 0.707\frac{C_t}{K_{ce}}.$$

Usually $C_t < K_{ce}$ so that PCM's are less damped than VCM's and often require intentional leakage paths to increase damping and ensure stability. Finally, we note the following analogy between VCM and PCM:

$$
\begin{array}{ccc}
(\mathrm{VCM}) & & (\mathrm{PCM}) \\
x_v & \longrightarrow & \theta \\
K_q & \longrightarrow & K_p N_p \\
P_L & \longrightarrow & P_r \\
K_{ce} & \longrightarrow & C_t
\end{array}
$$

## 4.6 Nonlinear Aspects

Consider the VCP system, for demonstration. Normally, the linearized valve expression

$$Q_L = K_q x_v - K_c P_L$$

is used where $K_q, K_c$ are evaluated at the null point,

$$x_v = 0; \quad P_L = 0; \quad Q_L = 0.$$

For large deviations from nominal, a nonlinear expression must be employed. For a critical center valve

$$Q_L = C_d \sqrt{\frac{P_S}{\rho}} w x_v \sqrt{1 - \frac{x_v}{|x_v|} \cdot \frac{P_L}{P_S}}.$$

Continuity gives,

$$Q_L = A_p v_p + C_{tp} P_L + \frac{V_t}{4\beta_e} \cdot \frac{dP_L}{dt}.$$

The simplified equation of motion for the piston, assuming inertial load only, is

$$P_L A_p = M_t \dot{v}_p.$$

Substituting,

$$\frac{C_d}{A_p} \sqrt{\frac{P_S}{\rho}} w x_v \sqrt{1 - \frac{x_v}{|x_v|} \cdot \frac{M_t}{P_S A_p} \cdot \frac{dv_p}{dt}} = \frac{V_t M_t}{4\beta_e^2 A_p^2} \cdot \frac{d^2 v_p}{dt^2} + \frac{C_{tp} M_t}{A_p^2} \cdot \frac{dv_p}{dt} + v_p$$

a nonlinear ODE describing the VCP combination. If $P_L/P_S$ is small,

$$\sqrt{1 - \frac{x_v}{|x_v|} \cdot \frac{P_L}{P_S}} \approx 1 - \frac{1}{2} \cdot \frac{x_v}{|x_v|} \cdot \frac{P_L}{P_S};$$

which is about 10% off for $P_L/P_S = 0.6$. Using this simplification, the equation becomes

$$\frac{1}{\omega_h^2} \cdot \frac{d^2 v_p}{dt^2} + \frac{2\delta_h}{\omega_h} \cdot \frac{dv_p}{dt} + v_p = \frac{C_d w}{A_p} \sqrt{\frac{P_S}{\rho}} x_v;$$

where

$$
\begin{aligned}
\omega_h &= \sqrt{\frac{4\beta_e A_p^2}{V_t M_t}}\,; \\
\delta_h &= \left(\frac{C_d w |x_v|}{2P_s}\sqrt{\frac{P_s}{\rho}} + C_{tp}\right)\left(\frac{M_t \omega_h}{2A_p^2}\right):
\end{aligned}
$$

Thus, although $\delta_h$ depends on $x_v$; i.e., the operating point, the hydraulic natural frequency remains the same. These expressions can be used for operation away from null.

# 5 ELECTROHYDRAULIC SERVOVALVES

## 5.1 Introduction

Hydraulic actuators are ideal for generating power output; when it comes to signal manipulation and feedback measurement, though, electrical devices are usually the choice. The connection between hydraulic actuators and electric devices is done through the electrohydraulic servovalve. Its function is to convert low power electrical signals into motion of a valve which controls flow to a hydraulic actuator. We have two main types of servovalves:

- Single–stage servovalve: a torque directly positions a spool valve.
- Two–stage servovavlve: a flapper valve is used as a first stage pre-amp, and a spool valve as a second stage.

According to the type of feedback used, we have:

- spool position feedback,
- load pressure feedback,
- load flow feedback.

## 5.2 Permanent Magnet Torque Motors

A permanent magnet torque motor, schematically shown in Figure 36, is the most popular device for stroking servovalves from an electrical signal. The torque or force produced is proportional to the input current. We want to develop the torque motor transfer function between,

$$
\begin{aligned}
x &= \text{output}\,; \\
e_g &= \text{input}:
\end{aligned}
$$

For the input amplifier we have,

$$e_1 = e_2 = \mu e_g \,;$$

where

$$\mu = \text{amplifier gain}\,.$$

For the armature coils,

$$e_1 + e_2 = 2\mu e_g = (R_c + r_p)\Delta i + \frac{2N_c}{10^8}\Delta\frac{d\phi_a}{dt}\,;$$

where

$$\begin{aligned}
R_c &= \text{coil resistance}\,;\\
r_p &= \text{any internal resistance}\,;\\
\Delta i &= i_1 - i_2\,;\\
\frac{2N_c}{10^8}\Delta\frac{d\phi_a}{dt} &= \text{induced voltage due to current flow}\\
&\phantom{=}\ \text{in the moving armature}\,;\\
\phi_a &= \text{total magnetic flux through the armature}\,.
\end{aligned}$$

As an aside, we remind that "Ohm's Law" for a magnetic circuit is

$$\mathcal{M} = \phi\mathcal{R}\,;$$

where

$$\begin{aligned}
\mathcal{M} &= \text{force or moment}\,;\\
\phi &= \text{magnetic flux}\,;\\
\mathcal{R} &= \text{reluctance}\,.
\end{aligned}$$

The armature flux is,

$$\phi_a = 2\phi_g\frac{x}{g} + \frac{N_c}{\mathcal{R}_g}\Delta i\,;$$

where

$$\begin{aligned}
\phi_g &= \text{flux in each gap with armature at neutral}\\
&\phantom{=}\ \text{(permanent flux)}\,;\\
\frac{N_c}{\mathcal{R}_g} &= \text{flux due to current flow in windings}\,.
\end{aligned}$$

Substituting:

$$2\mu e_g = (R_c + r_p)\Delta i + 2K_b s\theta + 2L_c s\Delta i\,;$$

where,

$$\begin{aligned}
\theta &= \frac{x}{a}\,;\\
K_p &= 2\times 10^{-8}\frac{a}{g}N_c\phi_g\,;\\
L_c &= 10^{-8}\frac{N_c^2}{\mathcal{R}_g}\,.
\end{aligned}$$

The armature torque is,

$$T_d = K_t \Delta i + K_m \theta;$$

where

$$\begin{aligned} K_t &= \text{armature torque constant;} \\ K_m &= \text{magnetic torque spring constant:} \end{aligned}$$

Mechanical torque balance gives,

$$T_d = J_a s^2 \theta + B_a s\theta + K_a \theta + T_L;$$

where the first three terms model the armature structure and the last term, $T_L$, is the output torque which is the product of the arm a and the valve stroking force. If we eliminate $T_d$ we get,

$$K_t \Delta i = J_a s^2 \theta + B_a s\theta + (K_a - K_m)\theta + T_L;$$

where we can see that the magnetic spring constant appears as a negative spring. If we assume the armature damping, $B_a$, is negligible and we eliminate $\Delta i$ we get the transfer function,

$$\theta = \frac{K_0 e_g - \dfrac{1}{K_a - K_m}\left(1 + \dfrac{s}{\omega_a}\right) T_L}{\dfrac{s^3}{\omega_a \omega_m^2 \left(1 - \dfrac{K_m}{K_a}\right)} + \dfrac{s^2}{\omega_m^2 \left(1 - \dfrac{K_m}{K_a}\right)} + \dfrac{s}{\omega_a \left(1 - \dfrac{K_m}{K_a}\right)} + 1};$$

where

$$K_0 = \frac{2K_t \mu}{(R_c + r_p)(K_a - K_m)};$$

is the static gain constant,

$$\omega_a = \frac{R_c + r_p}{2L_c};$$

is the armature circuit break frequency, and

$$\omega_m = \sqrt{\frac{K_a}{J_a}};$$

is the natural frequency of armature.

A simplified transfer function is,

$$\theta = \frac{K_0 e_g - \dfrac{1}{K_a - K_m}\left(1 + \dfrac{s}{\omega_a}\right) T_L}{\left(\dfrac{s}{\omega_r} + 1\right)\left(\dfrac{s^2}{\omega_0^2} + 2\dfrac{\delta_0}{\omega_0} s + 1\right)};$$

where

$$\begin{aligned} \omega_r &\approx \omega_a; \\ \omega_0 &\approx \omega_m; \\ \delta_0 &\approx \frac{1}{2} \cdot \frac{K_m}{K_a}: \end{aligned}$$

The transfer function between $\theta$ and $\Delta i$ is,

$$\frac{\theta}{\Delta i} = \frac{K_t}{J_a s^2 + B_a s + (K_a - K_m)}.$$

From the expression for $\theta$ we can see that the steady state stiffness of the torque motor to loads is in absolute value,

$$\left.\frac{\Delta T_L}{\Delta \theta}\right|_{\omega = 0} = K_a - K_m,$$

which is less than the mechanical stiffness $K_a$.

## 5.3 Single–Stage EHD Servovalves

In this case a torque motor is directly attached to a four–way spool valve. The spool valve is positioned by the torque motor and controls flow to a hydraulic actuator, as shown in Figure 37. Although flapper valves can also be used to form single–stage valves, they are not suitable for direct control of a load because of leakage characteristics. The stroking force is,

$$F_i = M_v \frac{d^2 x_v}{dt^2} + 0.43w(P_S - P_L)x_v,$$

where the first term is the inertia force and the second term is the steady–state flow force. If we linearize the last expression in $P_L$, $x_v$, we get

$$F_i = M_v s^2 x_v + 0.43w(P_S - P_{L_0})x_v - 0.43w x_{v_0} P_L.$$

If $r$ represents the radius arm of the torque motor,

$$x_v = r\theta,$$

and the stroking moment is given by

$$F_i r = M_v s^2 r^2 \theta + 0.43w r^2 (P_S - P_{L_0})\theta - 0.43rw x_{v_0} P_L.$$

If,

$J_a$ = armature inertia

$K_a$ = mechanical torsion spring constant of armature pivot

then the total torque developed on the armature due to current input is

$$T_d = (J_a + r^2 M_v)s^2\theta + [K_a + 0.43w r^2(P_S - P_{L_0})]\theta - 0.43rw x_{v_0} P_L.$$

The last term, $-0.43rw x_{v_0} P_L$ is the load torque $T_L$. Therefore, the transfer function is similar to the one produced in the previous section,

$$\left(\frac{s}{\omega_r} + 1\right)\left(\frac{s^2}{\omega_0^2} + 2\frac{\zeta_0}{\omega_0}s + 1\right)\theta = K_0 e_g - \frac{1}{K_{at}\left(1 - \frac{K_m}{K_{at}}\right)}\left(1 + \frac{s}{\omega_a}\right)T_L,$$

where,

$$
\begin{aligned}
K_{at} &= K_a + 0.43 r^2 w (P_S - P_{L0}) \\
K_0 &= \frac{2K_t \mu}{(R_c + r_p) K_{at} (1 - K_m / K_{at})} \\
\omega_m^2 &= \frac{K_{at}}{J_a + r^2 M_v}
\end{aligned}
$$

and $\omega_r$, $\omega_0$, and $\delta_0$ are the same as before.

Now assume that the valve controls a motor of displacement $D_m$ which, say. overcomes some inertia $J_t$. Then,

$$P_L D_m = J_t s^2 \theta_m \; .$$

Then the VCM transfer function is,

$$\frac{\theta_m}{x_v} = \frac{\dfrac{K_q}{D_m}}{s \left( \dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h} s + 1 \right)} \; .$$

Therefore, if

$$
\begin{aligned}
e_g &= \text{system input} \\
\theta_m &= \text{system output}
\end{aligned}
$$

the block diagram becomes as shown in Figure 38. We note that because of the positive feedback, the system may experience stability problems.

The open loop transfer function is:

$$GH = \frac{-K_1 s \left( 1 + \dfrac{s}{\omega_a} \right)}{\left( \dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h} s + 1 \right) \left( \dfrac{s^2}{\omega_0^2} + 2\dfrac{\delta_0}{\omega_0} s + 1 \right) \left( \dfrac{s}{\omega_r} + 1 \right)} \; ;$$

where

$$K_1 = \frac{0.43 r^2 w K_q x_{v0} J_t}{(K_{at} - K_m) D_m^2} \; ;$$

and the $-$ sign is used to convert the $(+, +)$ summing point into $(+, -)$. Since electronic responses are much faster than mechanical responses, $\omega_0 \gg \omega_h$. In addition, $\omega_r \approx \omega_a$. Therefore,

$$GH = \frac{-K_1 s}{\dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h} s + 1} \; .$$

The characteristic equation is

$$1 + GH = 0$$

or

$$\frac{s^2}{\omega_h^2} + \left(2\frac{\delta_h}{\omega_h} - K_1\right)s + 1 = 0,$$

and for stability we can see that we must have,

$$K_1 < 2\frac{\delta_h}{\omega_h}.$$

Now,

$$K_1 = \frac{0.43r^2 w K_q x_{v0} J_t}{(K_{at} - K_m)D_m^2} = \frac{0.43r^2 w x_{v0}}{K_{at} - K_m} \cdot \frac{K_q}{K_c} \cdot \frac{K_c J_t}{D_m^2},$$

and we have

$$\begin{aligned}
&\frac{K_q}{K_c} = \frac{2(P_S - P_{L0})}{x_{v0}} \qquad \text{(critical center valve)}\\
&\frac{K_c J_t}{D_m^2} = 2\frac{\delta_h}{\omega_h}\\
&0.43w(P_S - P_{L0}) \approx K_f\\
&K_{at} = K_a + r^2 K_f.
\end{aligned}$$

Then

$$K_1 = \frac{2r^2K_f}{K_a - K_m + r^2K_f} \cdot \frac{2\delta_h}{\omega_h},$$

and for stability

$$\frac{2r^2K_f}{K_a - K_m + r^2K_f} < 1,$$

or

$$\frac{r^2K_f}{K_a - K_m} < 1,$$

or

$$\frac{\text{(flow force spring rate)}}{\text{(net spring rate of torque motor)}} < 1.$$

In order to study the static performance of the servovalve, we assume steady state operation,

$$\begin{aligned}
T_d &= K_t \cdot i + K_m\theta,\\
T_L &= K_a\theta + 0.43wr^2(P_S - P_L)\theta.
\end{aligned}$$

Torque balance requires

$$T_d = T_L$$

and

$$\theta = \frac{x_v}{r} = \frac{K_t}{K_a + 0.43wr^2(P_S - P_L) - K_m} \cdot i.$$

The valve flow is

$$Q_L = C_d w x_v \sqrt{\frac{1}{\rho}(P_S - P_L)}\,;$$

or

$$Q_L = \frac{C_d w \sqrt{\frac{1}{\rho}(P_S - P_L)}\, r\, \Delta i}{\frac{K_a - K_m}{K_t}\left[1 + K_R\left(1 - \frac{P_L}{P_S}\right)\right]}\,;$$

where

$$K_R = \frac{0.43 w r^2 P_S}{K_a - K_m}\,.$$

Let $Q_L = Q_0$ when $\Delta i = \Delta i_{max}$, $P_L = 0$, and no flow force ($K_R = 0$):

$$Q_0 = \frac{r K_t C_d w\, \Delta i_{max}}{K_a - K_m}\sqrt{\frac{P_S}{\rho}}\,.$$

Then the flow–pressure curves for the single–stage servovalve are written as:

$$\frac{Q_L}{Q_0} = \frac{\sqrt{1 - \frac{P_L}{P_S}}}{1 + K_R\left(1 - \frac{P_L}{P_S}\right)} \cdot \frac{\Delta i}{\Delta i_{max}}\,.$$

## 5.4 Two–Stage Servovalve with Position Feedback

Single–stage servovalves are relatively simple and inexpensive but have two major faults. The flow capacity is limited because steady state flow forces on the spool tend to stall the torque motor and limit the valve stroke. The other disadvantage is the fact that stability depends to a large extent on the load dynamics. Although this can be minimized by proper servovalve design, each case should be investigated to assure stability. Two–stage servovalves overcome these disadvantages of limited flow capacity and instability. The most common type are two–stage servovalves with position feedback. This can be achieved in two basic ways: direct position feedback as shown in Figure 39, and force feedback where we use a spring to convert position to a force signal which is fed back to the torque motor.

Consider the two–stage servovalve with flapper–nozzle pilot stage and direct position feedback of Figure 39. The basic torque–motor transfer function is unchanged,

$$\theta = \frac{K_0 e_g - \frac{1}{K_{ae} - K_m}\left(1 + \frac{s}{\omega_a}\right) T_L}{\left(1 + \frac{s}{\omega_r}\right)\left(\frac{s^2}{\omega_0^2} + 2\frac{\delta_0}{\omega_0}s + 1\right)}\,;$$

where

$$K_{ae} = \text{mechanical plus flow force spring constant}$$
$$K_{ae} = K_a - r^2(8\pi C_{df}^2 P_S x_{f0})\,.$$

What we need is the transfer function

$$\frac{\theta_m}{x_f} = \frac{\theta_m}{r\theta}.$$

Torque motor/Flapper valve

The total force on the flapper is

$$\sum F = F_1 - F_2$$

and this is the force due to pressure inbalance plus the force due to jet deflection,

$$F_1 - F_2 = A_N P_{L_p} - 8\pi C_{df}^2 x_{f0} x_f P_S.$$

Consider a motion of the flapper to the left ($x_f > 0$). This causes the left hand flow to be restricted and the pressure differential $P_{L_p} = P_{1_p} - P_{2_p} > 0$. Therefore, the pressure force is restoring (to the right). Since the flapper nozzles are built into the valve, the net flapper motion is $(x_f - x_v)$ and this must replace $x_f$ in the above expression:

$$F_1 - F_2 = A_N P_{L_p} - 8\pi C_{df}^2 x_{f0} P_S (x_f - x_v).$$

The torque is

$$(F_1 - F_2)r = r A_N P_{L_p} + 8\pi r C_{df}^2 x_{f0} P_S x_v - 8\pi r C_{df}^2 x_{f0} P_S x_f,$$

where the first two terms in the right hand side ($r A_N P_{L_p} + 8\pi r C_{df}^2 x_{f0} P_S x_v$) represent $T_L$ and the last term is used to redefine $K_a$, the armature spring rate.

In order to go from the flapper to the spool valve:

$$\begin{aligned} x_v &= \text{output} \\ x_x - x_v &= \text{input.} \end{aligned}$$

The transfer function between $x_v$ and $x_f - x_v$ contains a quadratic term (due to the hydraulic natural frequency and damping ratio) and a first–order lag due to the flapper:

$$\frac{x_v}{x_f - x_v} = \frac{\frac{K_{qp}}{\omega_f A_v}}{\left(\frac{s}{\omega_f} + 1\right)\left(\frac{s^2}{\omega_{hp}^2} + 2\frac{\delta_{hp}}{\omega_{hp}} s + 1\right)}.$$

To go from $x_v$ to $P_{L_p}$, we use force balance:

$$P_{L_p} A_v = M_v \frac{d^2 x_v}{dt^2} + 0.43 w (P_S - P_L) x_v.$$

If we linearize around $P_{L_0} = 0$, we get

$$A_v P_{L_p} = M_v s^2 x_v + 0.43 w P_S x_v - 0.43 w x_{v_0} P_L.$$

Finally, we go to the hydraulic motor and the ineartial load,

$$P_L D_m = J_t s^2 \theta_m ;$$

$$\frac{\theta_m}{x_v} = \frac{\frac{K_q}{D_m}}{s\left(\frac{s^2}{\omega_h^2} + 2\frac{\delta_h}{\omega_h}s + 1\right)} .$$

Based on the above equations, we can draw the block diagram of the complete system as shown in Figure 40.

<u>Block diagram analysis</u>

There are two main feedback loops:

$$G_1 \ : \ \text{a spool positioning loop}$$

$$G_2 H_2 \ : \ \text{a pressure feedback loop} .$$

We have:

$$\frac{x_v}{x_f} = \frac{G_1}{1 + G_1} ;$$

$$G_1 = \frac{\frac{K_{qp}}{\omega_f A_v}}{\left(1 + \frac{s}{\omega_f}\right)\left(\frac{s^2}{\omega_{hp}^2} + 2\frac{\delta_{hp}}{\omega_{hp}}s + 1\right)} .$$

If $\omega_f \gg 1$ then,

$$G_1 = \frac{\frac{K_{qp}}{A_v}}{s\left(\frac{s^2}{\omega_{hp}^2} + 2\frac{\delta_{hp}}{\omega_{hp}}s + 1\right)} ;$$

and the requirement for stability is

$$\frac{K_{vp}}{\omega_{hp}} < 2\delta_{hp} ;$$

where

$$K_{vp} = \frac{K_{qp}}{A_v} ;$$

is the velocity static error coefficient.

If $\omega_f \gg \omega_{hp}$ but not zero, then the characteristic equation is

$$1 + G_1 = 0 ;$$

or

$$\frac{s^3}{\omega_{hp}^3} + \left(\frac{\omega_f}{\omega_{hp}^2} + 2\frac{\delta_{hp}}{\omega_{hp}}\right)s^2 + \left(1 + 2\frac{\delta_{hp}\omega_f}{\omega_{hp}}\right)s + (\omega_f + K_{vp}) = 0 .$$

The condition for stability is

$$\left(\frac{\omega_f}{\omega_{hp}^2} + 2\frac{\delta_{hp}}{\omega_{hp}}\right)\left(1 + 2\frac{\delta_{hp}\omega_f}{\omega_{hp}}\right) - \frac{1}{\omega_{hp}^2}(\omega_f + K_{vp}) > 0\,;$$

or

$$\frac{K_{vp}}{\omega_{hp}} + \frac{\omega_f}{\omega_{hp}} < \left(\frac{\omega_f}{\omega_{hp}} + 2\delta_{hp}\right)\left(1 + 2\delta_{hp}\frac{\omega_f}{\omega_{hp}}\right);$$

or

$$\frac{K_{vp}}{\omega_{hp}} < 2\delta_{hp}\left[1 + 2\delta_{hp}\frac{\omega_f}{\omega_{hp}} + \left(\frac{\omega_f}{\omega_{hp}}\right)^2\right];$$

and to first-order in $\omega_f/\omega_{hp}$,

$$\frac{K_{vp}}{\omega_{hp}} < 2\delta_{hp}\left(1 + 2\delta_{hp}\frac{\omega_f}{\omega_{hp}}\right).$$

Next we consider the pressure feedback loop. A reduced block diagram for this is shown in Figure 41. Simplification of this results in

$$\frac{G_1}{1+G_1} \approx \frac{1}{\left(\dfrac{s}{K_{vp}} + 1\right)\left[\dfrac{s^2}{\omega_{hp}^2} + \dfrac{1}{\omega_{hp}}\left(2\delta_{hp} - \dfrac{K_{vp}}{\omega_{hp}}\right)s + 1\right]};$$

which requires that

$$2\delta_{hp} \approx \frac{K_{vp}}{\omega_{hp}};$$

and

$$\frac{P_L}{x_v} \approx \frac{\dfrac{0.43wP_S}{A_v}\left(\dfrac{s^2}{\omega_h^2} - 2\dfrac{\delta_h}{\omega_h}s + 1\right)}{\dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h}s + 1};$$

which requires that

$$\omega_h \gg \frac{0.43wP_S}{M_v}.$$

If, furthermore,

$$K_{vp} \gg \omega_{hp};$$

then

$$\frac{G_1}{1+G_1} \approx \frac{1}{\dfrac{s}{K_{vp}} + 1};$$

and

$$G_2H_2 = \frac{\dfrac{r^2(8\pi C_f^2 P_S x_{f0} + r^2 A_N \frac{P_L}{x_v}}{K_{ie} - K_m}}{\left(\dfrac{s}{K_{vp}} + 1\right)\left(\dfrac{s^2}{\omega_0^2} + 2\dfrac{\delta_0}{\omega_0}s + 1\right)}.$$

The maximum value occurs near $\omega_0 \gg K_{vp}$, and

$$\frac{P_L}{x_v} \approx 0.43 w \frac{P_S}{A_v}.$$

The effect of the feedback loop may therefore be minimized by ensuring that:

$$|G_2 H_2|_{max} = \frac{r^2(8\pi C_f^2 P_S x_{f0}) + (A_N / A_v)(0.43 w r^2 P_S)}{K_{ae} - K_m} < 1.$$

With the pressure feedback loop thus minimized, we may approximate the servovalve by ($Q_L = K_q x_v$ for the no load, $P_L = 0$, flow):

$$\frac{Q_L}{e_g} = \frac{K_q x_v}{e_g} = \frac{K_q K_0 r}{\left(\frac{s}{\omega_r} + 1\right)\left(\frac{s^2}{\omega_0^2} + 2\frac{\delta_0}{\omega_0} s + 1\right)} \cdot \frac{G_1}{1 + G_1}$$

$$\approx \frac{r K_0 K_q}{\left(1 + \frac{s}{K_{vp}}\right)\left(\frac{s^2}{\omega_0^2} + 2\frac{\delta_0}{\omega_0} s + 1\right)};$$

for $\omega_r$ large.

<u>Steady State Performance</u>

We interpret the block diagram of Figure 40 for low frequency inputs. The torque motor is

$$\theta = K_0 e_g - \frac{T_L}{K_{ae} - K_m} = \frac{x_f}{r}.$$

The spool position is given by

$$x_v = \frac{x_f}{1 + \frac{A_v \omega_f}{K_{qp}}};$$

where

$$\omega_f = \frac{0.43 w P_S K_{cp}}{A_v^2};$$

and

$$\frac{A_v \omega_f}{K_{qp}} = \frac{0.43 w P_S}{A_v} \cdot \frac{K_{cp}}{K_{qp}} = \frac{0.43 w P_S}{A_v K_{pp}}.$$

By definition,

$$K_{pp} = \frac{P_S}{x_{f0}};$$

and

$$\frac{A_v \omega_f}{K_{qp}} = \frac{0.43 w x_{f0}}{A_v} = 4 \times 0.43 \frac{x_{f0}}{d_v}.$$

Since

$$x_{f0} \ll d_v;$$

it follows that at steady state

$$x_v \approx x_f .$$

The load torque is

$$T_L = r\left(8\pi C_{df}^2 P_S x_{f0} + \frac{A_N}{A_v}\cdot 0.43 w P_S\right) x_v .$$

If we combine the above equations and eliminate $T_L$, $x_f$, we can solve for

$$\frac{x_v}{e_g} = \frac{r K_0 (K_{ae} - K_m)}{(K_a - K_m) + r^2\left(\frac{A_N}{A_v}\, 0.43 w P_S\right)} ,$$

where we have substituted

$$K_{ae} = K_a - r^2 (8\pi C_{df}^2 P_S x_{f0}) .$$

The steady-state voltage to current ratio is

$$\frac{\Delta i}{e_g} = \frac{2\mu}{R_c + r_p} = \frac{K_0 (K_{ae} - K_m)}{K_t} .$$

If we use

$$K_R = 0.43 w r^2 \frac{P_S}{K_a - K_m} ,$$

we get

$$\begin{aligned}\frac{x_v}{\Delta i} = \frac{x_v}{e_g}\cdot\frac{e_g}{\Delta i} &= \frac{r K_t}{(K_a - K_m) + r^2\left(\frac{A_N}{A_v}\, 0.43 w P_S\right)} \\ &= \frac{r K_t}{(K_a - K_m)\left(1 + K_R \frac{A_N}{A_v}\right)} .\end{aligned}$$

If

$$K_R \frac{A_N}{A_v} \gg 1 ,$$

it follows that

$$\frac{x_v}{\Delta i} = \frac{r K_t}{K_a - K_m} .$$

The load flow is

$$Q_L = C_d w x_v \sqrt{\frac{1}{\rho}(P_S - P_L)} = \frac{C_d w}{\sqrt{\rho}}\left(\frac{x_v}{\Delta i}\right)\Delta i \sqrt{P_S - P_L} .$$

If we define

$$K_1 = \frac{C_d w}{\sqrt{\rho}}\left(\frac{x_v}{\Delta i}\right) ,$$

and

$$Q_{L\max} = K_1 \Delta i_{\max} \sqrt{P_S} ,$$

we get the equation for the flow–pressure curves

$$\frac{Q_L}{Q_{L_{max}}} = \frac{\Delta i}{\Delta i_{max}} \sqrt{1 - \frac{P_L}{P_S}}\,.$$

<u>Comparison with single state valve:</u>
We have

$$\frac{(Q_L)_{\text{single stage}}}{(Q_L)_{\text{two stage}}} = \frac{1 + K_R \frac{A_N}{A_v}}{1 + K_R \left(1 - \frac{P_L}{P_S}\right)}\,.$$

If

$$K_R \approx 1 \qquad \text{and} \qquad \frac{A_N}{A_v} \gg 1$$

it follows

$$\frac{(Q_L)_{\text{single stage}}}{(Q_L)_{\text{two stage}}} = \frac{1}{2 - \frac{P_L}{P_S}}\,.$$

Since $P_L < P_S$ we can see that

$$(Q_L)_{\text{single stage}} < (Q_L)_{\text{two stage}}\,;$$

which shows that single stage servovalves have limited flow capacity compared to two stage valves.

# 6 ELECTROHYDRAULIC SERVOMECHANISMS

A schematic representation of the material covered so far is shown in Figure 42. Incorporation of external feedback to the servovalve/VCM produces the so–called servomechanism, which is the subject of this chapter.

## 6.1 Design Considerations

1. <u>Supply Pressure</u>: Some of the relevant features are:

   - High pressure results in:
     - Low system specific weight.
     - Smaller trapped volumes.
     - High bulk modulus.
     - Better (faster) respnse.

    - Worse stability.

- Low pressure results in:
    - Low leakage.
    - Low thermal losses.
    - Low cost.
    - Low maintenance.

2. Power: Neglecting inefficiencies, the power $P$ is

$$P = P_L Q_L,$$

and

$$Q_L = C_d w x_v \sqrt{\frac{1}{\rho}(P_S - P_L)}.$$

Maximum power transfer to the load occurs at

$$P_L = \frac{2}{3} P_S.$$

We can see that we have no power in two cases:

- $P_L = 0$; i.e., all motion, no push ($P_1 = P_2 = \frac{1}{2} P_S$);
- $P_L = P_S$ or $Q_L = 0$; i.e., all push no motion ($P_1 = P_S$, $P_2 = 0$).

3. Actuator: It must be large enough to handle loads during operation. The hydraulic natural frequency must be large enough to avoid potential resonance.

4. Gear Ratio: Suppose we need a $10 in^3/rev$ displacement. There is a number of ways to achieve this. We can use:

- $10 in^3/rev$ motor with direct drive (gear ratio $n = 1$),
- $5 in^3/rev$ motor with $2:1$ gear ratio $n$,
- $2 in^3/rev$ motor with $5:1$ gear ratio $n$.

As $n$ is decreased:

- torque to inertia ratio is increased (the less inertia the beter),
- minimize nonlinear effects,
- better stiffness,

- lower operating speeds ⇒ better reliability.

As $n$ is increased:

- hydraulic natural frequency is increased,
- smaller motor ⇒ less cost.

> The best gear ratio is the smallest ratio which will give large enough (adequate) hydraulic natural frequency.

There are two basic configurations of electrohydraulic servomechanisms:

- position control, and
- velocity control.

## 6.2 Position Control Servos

The basic piece of additional information is the error signal derived from position feedback and generated by synchronous motors. There are, typically, two gains involved as shown in Figure 43,

$$\text{synchro gain} \quad K_e = \frac{e_s}{\theta_r - \theta_c};$$
$$\text{error amplifier gain} \quad K_d = \frac{e_g}{e_s}.$$

The complete block diagram is shown in Figure 44. We have the following individual transfer functions:

$$\frac{e_g}{\theta_e} = K_e K_d;$$

$$\frac{x_v}{e_g} = \frac{K_s}{\left(\frac{s}{\omega_1}+1\right)\left(\frac{s}{\omega_2}+1\right)\left(\frac{s^2}{\omega_0^2}+2\frac{\delta_0}{\omega_0}s+1\right)};$$

$$\theta_m = \frac{\frac{K_q}{D_m}x_v - \frac{K_{ce}}{D_m^2}\left(1+\frac{V_t}{4\beta_e K_{ce}}s\right)\frac{T_L}{n}}{s\left(\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1\right)};$$

$$\frac{\theta_c}{\theta_m} = \frac{1}{n}.$$

The open loop gain function is

$$A_u = \frac{K_v}{s\left(\frac{s}{\omega_1}+1\right)\left(\frac{s}{\omega_2}+1\right)\left(\frac{s^2}{\omega_0^2}+2\frac{\delta_0}{\omega_0}s+1\right)\left(\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1\right)}:$$

Neglecting all resonances higher than $\omega_h$, and assuming that there are none lower,

$$A_u = \frac{K_v}{s\left(\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1\right)};$$

where

$$K_v = K_e K_d K_s \cdot \frac{K_q}{D_m} \cdot \frac{1}{n};$$

is the velocity error coefficient. Therefore, we have a type–1 system, with position error

$$e_p = 0;$$

and velocity error,

$$e_v = \frac{1}{K_v}:$$

The condition for stability can be easily obtained,

$$K_v < 2\delta_h \omega_h:$$

The response of the closed–loop position control system is:

$$\frac{\theta_c}{\theta_r} = \frac{A_u}{1+A_u} = \frac{1}{\frac{s}{K_v}\left(\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1\right)+1};$$

or

$$\frac{\theta_c}{\theta_r} \approx \frac{1}{\left(\frac{s}{\omega_b}+1\right)\left(\frac{s^2}{\omega_c'^2}+2\frac{\delta_c'}{\omega_c'}s+1\right)};$$

where

$$\omega_b \approx K_v;$$
$$\omega_c' \approx \omega_h;$$
$$\delta_c' \approx \delta_h - \frac{K_v}{2\omega_h}:$$

<u>Bandwidth:</u> Usually defined as the frequency at which the amplitude ratio falls to 0.707 (3 db down) of its low frequency value,

$$\left|\frac{\theta_c}{\theta_r}\right| = \frac{1}{\sqrt{2}} = -3\ \text{db}:$$

In our case,

$$\left|\frac{\theta_c}{\theta_r}\right| = \frac{1}{\left[1+\left(\frac{\omega}{\omega_b}\right)^2\right]^{1/2}\left\{\left[1-\left(\frac{\omega}{\omega_{\eta c}}\right)^2\right]^2+\left(2\frac{\delta_{\eta c}\,\omega}{\omega_{\eta c}}\right)^2\right\}^{1/2}}:$$

For $\omega < \omega_b$, the response may be approximated by

$$\left|\frac{\theta_c}{\theta_r}\right| \approx \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^2}};$$

and for the bandwidth,

$$\frac{1}{\sqrt{2}} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^2}};$$

or

$$\omega \approx \omega_b \approx K_v = K_e K_d K_s \frac{K_q}{n D_m};$$

which means that $\omega_b$ is the bandwidth. We can see in it the influence of several factors,

$K_e$ : synchro output gain;
$K_d$ : synchro amplifier;
$K_s$ : servovalve power amplifier;
$\frac{K_q}{D_m}$ : valve/motor gain constant;
$\frac{1}{n}$ : gear ratio effect:

## 6.3 Velocity Control Servos

Assuming that $\omega_h$ is the only dominant frequency, we can construct the approximate block diagram shown in Figure 45. The open loop transfer function is,

$$A_{vu} = \frac{K_0}{\frac{s^2}{\omega_h^2}+2\frac{\delta_h}{\omega_h}s+1}:$$

We can see that we have a type–0 system with position error coefficient,

$$e_p = \frac{1}{1+K_0};$$

where the open loop gain constant is

$$K_0 = K_p = K_e K_s K_t \frac{K_q}{D_m}:$$

A Bode diagram is shown in Figure 46. The magnitude is

$$|A_{vu}| = \frac{K_p}{\left\{\left[1 - \left(\frac{\omega}{\omega_h}\right)^2\right]^2 + \left(\frac{2\delta_h \omega}{\omega_h}\right)^2\right\}^{1/2}};$$

and the phase angle

$$\phi = -\tan^{-1} \frac{2\delta_h \dfrac{\omega}{\omega_h}}{1 - \left(\dfrac{\omega}{\omega_h}\right)^2}:$$

The system is stable only because loop dynamics are so simply represented. The phase margin is dangerously small, especially if $\delta_h$ is small. Other lags, such as those associated with the servovalve can easily destabilize the loop. Therefore:

> Electrohydraulic velocity control servos must always be compensated to ensure stability if operating about null.

The closed loop response is given by,

$$\frac{\dot{\theta}_m}{\dot{\theta}_{mr}} = \frac{A_{vu}}{1 + A_{vu}} = \frac{K_p}{\dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h}s + (K_p + 1)}:$$

The steady–state response to a step input is:

$$\left(\frac{\dot{\theta}_m}{\dot{\theta}_{mr}}\right)_{ss} = \frac{K_p}{K_p + 1};$$

and

$$\left(\frac{\dot{\theta}_e}{\dot{\theta}_{mr}}\right)_{ss} = 1 - \frac{K_p}{K_p + 1} = \frac{1}{K_p + 1}:$$

Note that unless $K_p$ is very large (which is prohibited for stability reasons), there is always a steady–state offset given by $1/(K_p + 1)$. This offset depends upon $K_q$ which, in turn depends upon the operating point. Compensation, is therefore needed.

Compensation may also be needed in position control servos, as Figure 47 demonstrates. If the resonant peak of the quadratic rises above unity gain, then the system becomes unstable since the critical point of the Nyquist diagram would be encircled. Even if stability were not an issue, compensation would be highly desirable to raise the value of $K_v$ so that steady–state error is reduced.

## 6.4 Compensation

Compensation is often used in servomechanisms to increase low frequency gain or, as in velocity control servos, to decrease low frequency gain to ensure stability. A common method is to introduce a lag compensation network at an appropriate location in the loop.

Such networks are constructed based on the schematic electrical network of Figure 48. The relevant equations are:

$$
\begin{aligned}
e_i - e_o &= (\alpha - 1)R\, i ;\\
e_o &= iR + \frac{1}{C}\int i\, dt ;
\end{aligned}
$$

or

$$\frac{e_o}{e_i} = \frac{1 + RCs}{1 + \alpha RCs} .$$

Defining

$$\omega_{rc} = \frac{1}{RC} ;$$

we get

$$\frac{e_o}{e_i} = \frac{1 + \dfrac{s}{\omega_{rc}}}{1 + \dfrac{s}{(\omega_{rc}/\alpha)}} .$$

This network is called:

- lag element if $\alpha > 1$,
- lead element if $\alpha < 1$,

where

$$
\begin{aligned}
\alpha &= \text{lag to lead ratio} ;\\
\omega_{rc} &= \text{lead corner frequency} .
\end{aligned}
$$

To see this consider the phase angle of the element,

$$\phi = \tan^{-1}\left(\frac{\omega}{\omega_{rc}}\right) - \tan^{-1}\left(\frac{\omega\alpha}{\omega_{rc}}\right) ;$$

and observe that it is positive if $\alpha < 1$ and negative if $\alpha > 1$.

For the position servo, the compensated loop gain is

$$A_c(s) = \frac{K_{vc}\left(1 + \dfrac{s}{\omega_{rc}}\right)}{s\left(1 + \dfrac{s}{\omega_{rc}}\alpha\right)\left(\dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h}s + 1\right)} ;$$

where

$$K_{vc} = \alpha K_v ;$$

is the compensated velocity coefficient. Quantities $\omega_h$, $\delta_h$ are fixed. We need to choose $\alpha$, $\omega_c$, $K_{vc}$, and $\omega_{rc}$. We can do this as follows:

1. Determine the frequency between $\omega_{rc}$ and $\omega_h$ where the phase lag is minimum. The gain crossover frequency is $\omega_c$ and we can obtain the maximum phase margin.

2. Adjust $\omega_{rc}$ to get adequate phase margin; 50 or 60 degrees will do.

3. Choose $\alpha$ to produce adequate $K_{vc}$ for acceptable steady–state error. Practical considerations limit usually the value of $\alpha$ to about 10 or so.

## 6.5 Compensation for Stability

The goal of the compensation in this case is to increase stability, or bring the gain crossover frequency down to a value below $\omega_h$. For this a pure lag network, Figure 50, is sufficient. The relevant equations are,

$$\frac{e_o}{e_i} = \frac{1}{1 + \dfrac{s}{(\omega_{rc}/\alpha)}},$$

where

$$\omega_{rc} = \frac{1}{RC},$$

or

$$\frac{e_o}{e_i} = \frac{1}{1 + T_c s},$$

where

$$T_c = \frac{\alpha}{\omega_{rc}}.$$

For a system with gain constant $K_p$, the corner frequency is determined by

$$\omega_b = \frac{1}{T_c} = \frac{\omega_c}{K_p},$$

where $\omega_c$ is the desired gain crossover frequency, $\omega_c < \omega_h$. Computing $T_c$ and by fixing $\alpha$ to between 10 and 20, the values of $R$ and $C$ can be chosen. The loop gain becomes,

$$A_{vc} = \frac{K_p}{(1 + T_c s)\left(\dfrac{s^2}{\omega_h^2} + 2\dfrac{\delta_h}{\omega_h}s + 1\right)},$$

as illustrated in Figure 51.

If Routh's criterion is applied to the characteristic equation of the closed loop compensated system

$$1 + A_{vc} = 0,$$

the criterion for stability is

$$(1 + K_p) < \left(1 + \frac{1}{T_c} \cdot \frac{2\delta_h}{\omega_h}\right)(1 + 2\delta_h \omega_h T_c),$$

and, if $2\delta_h T_c \omega_h \gg 1$, this reduces to

$$K_p < 2\delta_h \omega_h T_c ;$$

or, with the crossover frequency given by

$$\omega_c = \frac{K_p}{T_c} ;$$

the stability condition becomes

$$\omega_c < 2\delta_h \omega_h :$$

The corner frequency of the lag is therefore given by

$$\omega_b = \frac{\omega_c}{K_p} < \frac{2\delta_h \omega_h}{K_p} :$$

With typical values of $\delta_h = 0.1$ to $0.2$ (near null) the corner frequency is in the range $\omega_b < (0.2$ to $0.4)\omega_h/K_p$ with a crossover frequency margin given by $\omega_c < (0.2$ to $0.4)\omega_h$. Note that the above analysis becomes exact if the first order lag is replaced by a pure integrator $1/T_c s$.

## 6.6 Gear Ratios in Rotary Drives

The purpose of this section is to show that:

> As the gear ratio $n$ is increased, the ratio of torque to inertia at the load is decreased and the hydraulic natural frequency is increased.

Using the configuration shown in Figure 52, we have:

$$\begin{aligned} T_m &= J_m \ddot{\theta}_m + F_t r_p ; \\ F_t r_g &= T_L + J_L \ddot{\theta}_L ; \end{aligned}$$

where $F_t$ represents the contact force between the two drives. Therefore,

$$T_m = J_m \ddot{\theta}_m + \frac{r_p}{r_g}(T_L + J_L \ddot{\theta}_L) :$$

If we denote

$$\frac{r_p}{r_g} = \frac{\theta_L}{\theta_m} = \frac{1}{n} ;$$

then

$$T_m = \left( J_m + \frac{J_L}{n^2} \right) \ddot{\theta}_m + \frac{T_L}{n} ;$$

where

$$\begin{aligned} \frac{J_L}{n^2} &= \text{load torque reflected to motor shaft} \\ \frac{T_L}{n} &= \text{inertia of load reflected to motor shaft:} \end{aligned}$$

We can also write:

$$(J_m n^2 + J_L)\frac{\ddot{\theta}_m}{n^2} + \frac{T_L}{n} = T_m \; ;$$

or

$$(J_m n^2 + J_L)\ddot{\theta}_L + T_L = nT_m \; ;$$

where

$J_m n^2$ = inertia of motor reflected to output shaft

$nT_m$ = motor output torque referred to output shaft.

For maximum acceleration of the load, we must maximize the ratio of torque to inertia at the load, i.e.,

$$\text{maximize} \quad \frac{nT_m}{J_m n^2 + J_L} \; ;$$

or, since for a given load and speed, $nT_m$ is constant,

$$\text{minimize} \quad n^2 J_m \; .$$

Let the subscript G denote geared and D direct drive. The motor inertia is empirically shown to be

$$J_m \propto \left(\frac{1}{n}\right)^{1.5} .$$

Therefore,

$$\frac{n^2 (I_m)_G}{(J_m)_D} = n^2 \left(\frac{1}{n}\right)^{1.5} = n^{0.5} \; .$$

It follows then that as gear ratio increases, load acceleration decreases.

The inertia to be used in the hydraulic natural frequency is the motor inertia plus the load inertia reflected to the motor shaft. Recall that,

$$\omega_h = \left(\frac{4\beta_e D_m^2}{V_t J_t}\right)^{1/2} .$$

Now

$$V_t \propto D_m \; ;$$

and

$$J_t = J_m + \frac{J_L}{n^2} \; ;$$

or, since

$$J_m \propto \left(\frac{1}{n}\right)^{1.5} \propto D_m^{1.5} \; ;$$

we have

$$J_t = K D_m^{1.5} + \frac{J_L}{n^2} \; ;$$

and

$$\omega_h \approx \left[ \frac{D_m^2}{D_m \left( K D_m^{1.5} + \frac{J_L}{n^2} \right)} \right]^{1/2} .$$

For a given speed at the load, $D_m \approx 1/n$, so that

$$\omega_h \approx \left( \frac{1}{\frac{K_1}{n^{0.5}} + K_2 \frac{J_L}{n}} \right)^{1/2} .$$

Therefore, the hydraulic natural frequency increases as $n$ is increased.

The expression above can be further manipulated to give,

$$\omega_h \approx \frac{n^{1.25}}{\left( n^2 + \frac{J_L}{J_m} \right)^{1/2}} = \frac{n^{0.5}}{\left( n^{0.5} + \frac{J_L}{J_{mD}} \right)^{1/2}} ,$$

or

$$\frac{\omega_h}{(\omega_h)_D} = n^{0.5} \left( \frac{1 + \frac{J_L}{J_{mD}}}{n^{0.5} + \frac{J_L}{J_{mD}}} \right)^{1/2} ,$$

where $J_{mD}$ is the motor inertia with direct drive. Thus, if

$$J_L \gg J_{mD} \qquad \Rightarrow \qquad \frac{\omega_h}{(\omega_h)_D} = n^{0.5} ,$$

and if

$$J_L \ll J_{mD} \qquad \Rightarrow \qquad \frac{\omega_h}{(\omega_h)_D} = n^{0.25} .$$

## 6.7 Summary of EHD Position Control Servo

1. Close loop with position feedback.
2. Stability.
   (a) Establish servo loop transfer function, $A_u$.
   (b) Approximate $A_u$ in terms of lowest resonance, $\omega_h$.
   (c) Establish approximate design criterion for loop stability, $K < 2\zeta_h \omega_h$.
3. System performance.
   (a) Closed loop response, $\frac{\theta_c}{\theta_r}$ = a cubic.
   (b) Approximate cubic as lag at $\omega_b$ plus quadratic term.
   (c) Determine bandwidth based on lowest corner frequency, $\omega_b$.
4. Compensation, as shown in Figures 53 through 55.

There are some general criteria applicable to the "good" design of any servo system.

1. There must always be a range of frequencies where the loop gain is substantially greater than unity. All the desirable characteristics of feedback control are based on this simple fact. When the loop gain is less than unity, feedback is not effective and the loop is essentially open. The crossover frequency gives the borderline between open loop and closed loop control.
2. There is always an accuracy requirement and this necessitates some loop gain greater than unity.
3. A system should never be designed conditionally stable unless it cannot be avoided.
4. For satisfactory stability, the crossover frequency should occur on an asymptotic $-1$ slope and it must be "controlled". That is, establishment of the crossover frequency must be an explicit part of design, and its value and variation must be computed to assure stability under all operating conditions.
5. Noise rejection and stability always limit the system bandwidth. In fact it is desirable to keep the bandwidth at a minimum consistent with specifications. A reduced bandwidth usually simplifies compensation and, because peak power outputs are associated with high frequencies, relaxes requirements on individual elements, thereby producing savings in cost.
6. Accuracy requirements usually dictate the slope of the Bode diagram at low frequencies, that is, zero for type 0 systems, $-1$ for type 1 systems, $-2$ for type 2 systems, and so on.

These constraints, or good design features, are represented in the Bode diagram of Figure 56.

# 7 SPECIAL TOPICS

## 7.1 Pressure Transients in Fluid Power Control Systems

Consider a simple mass–spring system. The governing equation is simply Newton's law:

$$m\frac{d^2x}{dt^2} = -kx\,.$$

Integrating, we get

$$m\frac{du}{dt} = -kx \Rightarrow m\frac{du}{dx}\cdot\frac{dx}{dt} = -kx \Rightarrow mu\,du = -kx\,dx \Rightarrow$$
$$\frac{mu^2}{2} + \frac{kx^2}{2} = \text{const.} = (\text{kinetic energy developed}) + (\text{energy stored})\,.$$

If $u_0$ be the velocity at $x = 0$ and $x_0$ the position when $u = 0$, then

$$\frac{mu_0^2}{2} + 0 = 0 + \frac{kx_0^2}{2}.$$

Now draw an analogy with the "trapped fluid spring" of volume $V_0$, i.e.,

$$K_h = \frac{\beta_e A_p^2}{V_0}.$$

We have,

$$\frac{1}{2}M_t v_{p0}^2 = \frac{1}{2}K_h x_{p0}^2.$$

The maximum pressure in the trapped fluid spring will occur when the piston velocity is zero. A force balance under this condition gives,

$$P_{2max} A_p = K_h x_{p0}.$$

Eliminating $x_{p0}$, we get

$$P_{2max} = \frac{v_{p0}}{A_p^2}\sqrt{K_h M_t} = v_{p0}\sqrt{\frac{\beta_e M_t}{V_0}}.$$

This expression neglects the effects of any damping present in the system.

<u>2nd Approximation</u>
Consider the VCP with inertia load only,

$$Q_L = A_p s x_p + \frac{V_t}{4\beta_e} s P_L;$$

$$P_L A_p = M_t s^2 x_p.$$

If we combine and eliminate $x_p$, we get

$$P_L = \frac{4\beta_e}{V_t} \cdot \frac{sQ_L}{s^2 + \omega_h^2};$$

where

$$\omega_h^2 = \frac{4\beta_e A_p^2}{M_t V_t}.$$

As the control valve is closed, with $Q_L$ held constant (constant piston speed), $P_1$ will decrease and $P_2$ will increase until at the instant of valve closure, $P_1 = P_2 = P_S/2$ and $Q_L$ will decrease in a step change from $Q_L = A_p s x_p$ to $Q_L = 0$. Let this instant define $t = 0$, $x_p = x_{p0}$, $sx_p = v_{p0}$.

In the Laplace domain the step change in $Q_L$ is given by

$$Q_L(s) = -\frac{A_p v_{p0}}{s};$$

and the pressure response is

$$P_L = \frac{\frac{4\beta_e s}{V_t}\left(-\frac{A_p v_{p0}}{s}\right)}{s^2+\omega_h^2} = -\frac{4\beta_e v_{p0}}{\omega_h V_t}\cdot\frac{\omega_h}{s^2+\omega_h^2}:$$

Inverting to the time domain, with $P_L(0) = 0$,

$$P_L(t) = -\frac{4\beta_e v_{p0}}{\omega_h V_t}\sin\omega_h t:$$

This expression can only be valid until $t = t_1$ when $P_1$ approaches zero. Define $P_R$ as the ratio of the approximate peak pressure to supply pressure,

$$P_R = \frac{v_{p0}}{P_S}\sqrt{\frac{\beta_e M_t}{V_0}};$$

and

$$P_L(t) = -\sqrt{2}P_S P_R \sin\omega_h t:$$

Assuming that $P_1$ and $P_2$ continue a symmetrical divergence until $P_1 \to 0$, the time $t = t_1$ can be found,

$$-P_L(t_1) = P_2(t_1) - P_1(t_1) = P_S - 0;$$

or

$$-P_S = -\sqrt{2}P_S P_R \sin\omega_h t_1;$$

or

$$t_1 = \frac{1}{\omega_h}\sin^{-1}\left(\frac{1}{\sqrt{2}P_R}\right):$$

The rate of change of the load pressure drop is,

$$\dot{P}_L = -\sqrt{2}\omega_h P_S P_R \cos\omega_h t:$$

At $t = t_1$ we get

$$\dot{P}_L = \dot{P}_1 - \dot{P}_2 = -2\dot{P}_2;$$

and symmetry is assumed, see Figure 57. Then

$$\dot{P}_2(t_1) = \frac{\sqrt{2}}{2}\omega_h P_S P_R \cos\left[\sin^{-1}\left(\frac{1}{\sqrt{2}P_R}\right)\right];$$

or

$$\dot{P}_2(t_1) = \frac{\omega_h P_S}{2}\sqrt{2P_R^2 - 1}:$$

For times beyond $t_1$, $P_2$ continues to increase in excess of $P_S$. Further analysis begins with an initial condition of $P_2(t_1) = P_S$ and $\dot{P}(t_1)$ given above.

The supply chamber is assumed to remain at zero while $P_2 > P_S$ and, during this period, damping due to leakages is taken into account. Thus,

$$A_p v_p = Q_L = \frac{V_0}{\beta_e}\cdot\frac{dP_2}{dt} + (C_{ip} + C_{ep} + K_c)P_2;$$

and

$$P_2 A_p = -M_t s^2 x_p = -M_t s v_p :$$

Eliminating $v_p$:

$$\left( \frac{M_t V_0}{\beta_e A_p} s^2 + \frac{K_{ce} M_t}{A_p^2} s + 1 \right) P_2(s) = \text{terms due to initial conditions} ;$$

where we have denoted

$$K_{ce} = C_{ip} + C_{ep} + K_c :$$

If

$$\omega_2 = \frac{\omega_h}{\sqrt{2}} ; \qquad \delta_2 = \frac{1}{2} \cdot \frac{K_{ce}}{A_p} \sqrt{\frac{\beta_e M_t}{V_0}} ;$$

then

$$\left( \frac{s^2}{\omega_2^2} + 2\frac{\delta_2}{\omega_2} s + 1 \right) P_2(s) = I ;$$

where

$$I = \frac{1}{\omega_2^2} \left[ (2\delta_2 \omega_2 + s) P_2(0) + \dot{P}_2(0) \right] :$$

The system is thus second order with "input" arising from the initial pressure and its rate of change. The solution in the time domain is

$$\frac{P_2(t)}{P_S} = \frac{e^{-\delta_2 \omega_2 \Delta t}}{\sqrt{1-\delta_2^2}} \left[ \left( \sqrt{P_R^2 - \frac{1}{2}} + \delta_2 \right) \sin\left( \omega_2 \sqrt{1-\delta_2^2}\, \Delta t \right) + \sqrt{1-\delta_2^2} \cos\left( \omega_2 \sqrt{1-\delta_2^2}\, \Delta t \right) \right] ;$$

where

$$\Delta t = t - t_1 > 0 :$$

The peak occurs at

$$\Delta t_m = t_2 - t_1 ;$$

where

$$\omega_2 \Delta t_m = \frac{1}{\sqrt{1-\delta_2^2}} \tan^{-1} \left[ \frac{\sqrt{(1-\delta_2^2)\left(P_R^2 - \frac{1}{2}\right)}}{1 + \delta_2 \sqrt{P_R^2 - \frac{1}{2}}} \right] ;$$

and is given by

$$\frac{P_{2max}}{P_S} = e^{-\delta_2 \omega_2 \Delta t_m} \sqrt{\frac{1}{2} + P_R^2 + 2\delta_2 \sqrt{P_R^2 - \frac{1}{2}}} ; \quad \text{for} \quad \delta_2 < 1 :$$

For small values of the damping ratio $\delta_2$ and large values of $P_R$,

$$\omega_2 \Delta t_m \approx \tan^{-1}(\infty) = \frac{\pi}{2} ;$$

and

$$\frac{P_{2_{max}}}{P_S} = P_R e^{\frac{-\pi\delta_2}{2}} .$$

For small value of peak pressure we want large $\delta_2$, which means that we have to employ a relief valve.

<u>Relief Valves:</u> The presence of a relief valve in the system will lead to values of $\delta_2 > 1$. In this case the maximum pressure $P_{2_{max}}$ is shown in Figure 58. The response is not oscillatory and the maximum pressure is evaluated as $\Delta t \to 1$. When this expression is evaluated for $\delta_2 \gg 1$, the result is

$$\frac{P_{2_{max}}}{P_S} \approx 1 + \frac{P_R}{2\delta_2} .$$

The value of $\delta_2$ is given by

$$\delta_2 = \frac{K_r + K_c + C_{ip} + C_{ep}}{2A_p} \cdot \sqrt{\frac{\beta_e M_t}{V_0}} ,$$

where $K_r$ is the coefficient of relief flow,

$$Q_r = K_r(P_2 - P_S) ,$$

and the valve is set to open at $P_S$. The coefficient $K_r$ is intentionally large relative to the leakage coefficients, and

$$\delta_2 \approx \frac{K_r}{2A_p}\sqrt{\frac{\beta_e M_t}{V_0}} .$$

The maximum relief flow is when $P_2 = P_{2_{max}}$, so that

$$Q_{r_{max}} = K_r(P_{2_{max}} - P_S) = \frac{K_r P_R P_S}{2\delta_2} ,$$

or

$$Q_{r_{max}} = A_p \left(\frac{\beta_e M_t}{V_0}\right)^{-1/2} P_R P_S = A_p v_{p0} .$$

This relationship is useful in estimating the necessary flow capacity of relief valves.

For a rotary system, the equivalent expression is

$$Q_{r_{max}} = D_m \dot{\theta}_m ,$$

and

$$P_R = \frac{\dot{\theta}_{m0}}{P_S}\sqrt{\frac{\beta_e J_t}{V_0}} ,$$

where $\dot{\theta}_{m0}$ is the motor speed at the "sudden stoppage design point."

## 7.2 Hydraulic Power Transmission

1. Introduction

The connection between the power–generating, power–controlling, and power–utilization devices requires the transmission of flows and pressures through the transmission lines. Since change in fluid power requires pressure changes, transmission of pressure signals becomes an important consideration in system design to assure dynamic stability and speed of response.

2. Steady flow

Hydraulic circuits are characterized by a number of bends in tubing and various fittings. The total pressure drop in a system is,

$$\Delta P = \sum_{i=1}^{n_t} f_i \frac{L_i}{D_i} \cdot \frac{\rho V_i^2}{2} + \sum_{i=1}^{n_f} f_i \frac{L_{ei}}{D_i} \cdot \frac{\rho V_i^2}{2} + \sum_{i=1}^{n_b} f_i \frac{L_{ei}}{D_i} \cdot \frac{\rho V_i^2}{2};$$

where

$$\begin{aligned} n_t &= \text{number of tubes} \\ n_f &= \text{number of fittings} \\ n_b &= \text{number of bends} \\ L_e &= \text{equivalent length for each bend and tubing.} \end{aligned}$$

$f$ is a friction factor given by,

$$f = \frac{64}{Re};$$

for $Re < 2000$, and

$$\frac{1}{\sqrt{f}} = 2\log_{10}(Re \cdot \sqrt{f}) - 0.8;$$

for $Re > 4000$.

Velocities in hydraulic circuits are normally limited due to practical considerations. Exceeding the recommended values means larger pressure losses and temperature rises. Weight and cost penalties result from velocities that are too low. Typical values are:

- Suction lines: 20–75 in/sec.
- Discharge lines: 100–200 in/sec.
- Flow in relief valves: 1000 in/sec.

3. Dynamic response of hydraulic transmission lines

With unsteady flow through the piping of a hydraulic system, fluid mass and compressibility effects can introduce undesirable transients and deterioration of system response. The natural frequencies of a transmission line of length $L$ are given by the organ pipe frequencies from classical physics and depending on the boundary conditions are:

$$f = \frac{2nC_0}{4L} = \frac{nC_0}{2L}; \quad n = 1, 2, 3, \ldots$$

or

$$f = \frac{(2n - 1)C_0}{4L};$$

where the speed of wave propagation is

$$C_0 = \sqrt{\frac{\beta_e}{\rho_0}};$$

and it generally lies between 35,000 and 50,000 in/sec.

When the line length is small compared to the wavelengths contained in the pressure and flow signals, a lumped model can simplify the analysis considerably. Referring to Figure 58, energy is accumulated according to

$$C\frac{dP}{dt} = \Delta Q;$$

and

$$I\frac{dQ}{dt} = \Delta P;$$

where

$$C = \frac{A^2}{k}; \quad \text{for a spring–backed piston accumulator}$$
$$I = \frac{\rho L}{A}; \quad \text{for inertial energy storage uniform velocity profile:}$$

For the three–lump model of Figure 59,

$$\begin{aligned}
\frac{dP_a}{dt} &= (Q_a - Q_1)\frac{3}{C}; \\
\frac{dP_1}{dt} &= (Q_1 - Q_2)\frac{3}{C}; \\
\frac{dP_2}{dt} &= (Q_2 - Q_b)\frac{3}{C}; \\
\frac{dQ_1}{dt} &= (P_a - P_1)\frac{3}{I}; \\
\frac{dQ_2}{dt} &= (P_1 - P_2)\frac{3}{I}; \\
\frac{dQ_b}{dt} &= (P_2 - P_b)\frac{3}{I};
\end{aligned}$$

which can be solved numerically with $Q_a, P_b$ as inputs and $P_a, Q_b$ as outputs. Such models require enough lumps for accurate representation of wave propagation effects. Usually, one must use about 10 lumps per shortest signal wavelength, where the wavelength $\lambda$ is related to the signal frequency $f$ by

$$\lambda = \frac{C_0}{f} .$$

If fluid properties are assumed to uniformly distributed, the continuity and momentum equations, assuming negligible friction and no nominal flow, give

$$\begin{aligned} -\frac{\partial V}{\partial x} &= \frac{1}{\beta_e} \cdot \frac{\partial P}{\partial t} ; \\ -\frac{\partial P}{\partial x} &= \rho \frac{\partial V}{\partial t} . \end{aligned}$$

These equations, when combined, form a second-order wave equation. The solution for the overpressure (excess pressure over the static pressure) in the s-domain is

$$P(x;s) = P^{+}(s)e^{-\Gamma x} + P^{-}(s)e^{+\Gamma x} ; \quad \Gamma = \frac{s}{C_0} ;$$

where $P^{+}$ represents a wave travelling in the forward direction while $P^{-}$ represents a pressure wave travelling in the reverse direction. $P^{+}$ and $P^{-}$ are established by the boundary conditions at the ends of the line.

The forward traveling wave is,

$$P(s;x) = P^{+} e^{-sx/C_0} ;$$

which is the transform of a pressure wave $P^{+}(t)$ delayed in time by the amount $x/C_0$. Thus, the delay time for the wave to travel down the entire line of length $L$ is $T$,

$$T = \frac{L}{C_0} .$$

Depending on the different ways in which a transmission line can be connected to other elements in a hydraulic system, we have the following four solutions of the equations:

$$\begin{bmatrix} P_a \\ P_b \end{bmatrix} = \begin{bmatrix} \frac{Z_0 \cosh(Ts)}{\sinh(Ts)} & \frac{-Z_0}{\sinh(Ts)} \\ \frac{Z_0}{\sinh(Ts)} & \frac{-Z_0 \cosh(Ts)}{\sinh(Ts)} \end{bmatrix} \begin{bmatrix} Q_a \\ Q_b \end{bmatrix} ;$$

$$\begin{bmatrix} P_a \\ Q_b \end{bmatrix} = \begin{bmatrix} \frac{Z_0 \sinh(Ts)}{\cosh(Ts)} & \frac{1}{\cosh(Ts)} \\ \frac{1}{\cosh(Ts)} & \frac{-\sinh(Ts)}{Z_0 \cosh(Ts)} \end{bmatrix} \begin{bmatrix} Q_a \\ P_b \end{bmatrix} ;$$

$$\begin{bmatrix} Q_a \\ P_b \end{bmatrix} = \begin{bmatrix} \frac{\sinh(Ts)}{Z_0 \cosh(Ts)} & \frac{1}{\cosh(Ts)} \\ \frac{1}{\cosh(Ts)} & \frac{-Z_0 \sinh(Ts)}{\cosh(Ts)} \end{bmatrix} \begin{bmatrix} P_a \\ Q_b \end{bmatrix} ;$$

$$\begin{bmatrix} Q_a \\ Q_b \end{bmatrix} = \begin{bmatrix} \frac{\cosh(Ts)}{Z_0 \sinh(Ts)} & \frac{-1}{Z_0 \sinh(Ts)} \\ \frac{1}{Z_0 \sinh(Ts)} & \frac{-\cosh(Ts)}{Z_0 \sinh(Ts)} \end{bmatrix} \begin{bmatrix} P_a \\ P_b \end{bmatrix} .$$

In deciding which case to use, it helps to view a valve with high flow gain, or a pump, as a flow input and an accumulator or large trapped volume of oil as a pressure input. A blocked end is a zero flow input and an open end is a zero pressure input. For example, for the transmission line of the lumped model of Figure 59, the second case of the above equations applies:

$$
\begin{aligned}
P_a(s) &= \frac{Z_0 \sinh(Ts)}{\cosh(Ts)} Q_a(s) + \frac{1}{\cosh(Ts)} P_b(s)\,; \\
Q_b(s) &= \frac{1}{\cosh(Ts)} Q_a(s) - \frac{\sinh(Ts)}{Z_0 \cosh(Ts)} P_b(s)\,;
\end{aligned}
$$

where

$$Z_0 = \frac{C_0}{A}$$

is the characteristic wave impedance.

Frequency response computations are easier with distributed models because of the equations,

$$
\begin{aligned}
\cosh(j\omega t) &= \cos(\omega t)\,; \\
\sinh(j\omega t) &= j \sin(\omega t)\,.
\end{aligned}
$$

Transient response computations are easier with differential equations. One way of reducing transfer functions to polynomials is to employ finite products,

$$\cosh(Ts) = \prod_{n=0}^{\infty} \left[ 1 + \left( \frac{s}{\omega_n} \right)^2 \right]; \quad \omega_n = \frac{(2n+1)\pi}{2T}\,;$$

and

$$\sinh(Ts) = (Ts) \prod_{m=1}^{\infty} \left[ 1 + \left( \frac{s}{\omega_m} \right)^2 \right]; \quad \omega_m = \frac{m\pi}{T}\,.$$

4. Friction effects

Fluid friction acts to damp out transmission line transients. There are two main friction models in use; the constant friction model which is simpler to use but it generally underestimates damping, and a frequency dependent model where the various damping ratios depend on the corresponding frequency.

## 7.3 Describing Function Analysis

1. Introduction

For nonlinear systems the principle of superposition of solutions does not hold. In general, the response of nonlinear systems will depend on both magnitude and type of input and it may be completely different for step inputs of different magnitude or sinusoidal inputs of different frequencies. The response may also depend drastically on the initial conditions. Some of the relevant phenomena are:

1. Frequency–amplitude dependence: Consider Duffing's equation, spring–mass–damper with a nonlinear spring,
$$m\ddot{x} + b\dot{x} + kx + k'x^3 = 0 .$$
Typical force–displacement curves are shown in Figure 61. We refer to $k' > 0$ as a hardening spring, $k' < 0$ as softening spring, while $k' = 0$ is naturally a linear spring. The natural motion (frequency of free oscillations) for the linear spring is $\sqrt{k/m}$, and this is constant; i.e., it does not depend on the amplitude of motion, see Figure 61. The equivalent spring constant (the slope of the spring force vs. displacement curve) for the nonlinear system is $k + 3k'x^2$, and we can see, as Figure 61 demonstrates, that in this case the natural frequency will depend on the amplitude of motion. A hardening spring will oscillate at higher frequencies at high amplitudes whereas the opposite is true for a softening spring.

2. Jump phenomena: Consider again Duffing's equation, this time adding sinusoidal forcing,
$$m\ddot{x} + b\dot{x} + kx + k'x^3 = P\cos\omega t .$$
The frequency response curve for $k' = 0$ has the familiar form shown in Figure 62. As Figure 61 suggests we can visualize the frequency response curves for $k' > 0$ and $k' < 0$ by bending the linear frequency response curve in the appropriate direction, so that it wraps around the natural motion curve, see Figure 62. We can see that as the excitation frequency is increasing or decreasing, the system may exhibit unstable oscillations or multiple–valued oscillations where the amplitude of motion will depend on the initial conditions.

3. Subharmonic oscillations: For excitation frequency $\omega$, a nonlinear system may experience responses, besides $\omega$, at frequencies $\omega/n$ where $n$ is an integer. These are called subharmonics. Superharminic oscillations, at frequencies $n\omega$, are also possible although not as severe as subharmonics. Generation of these oscillations depends upon initial conditions, as well as amplitude and frequency of excitation.

4. Limit cycles: Limit cycles are isolated, self–excited oscillations (i.e., in the absence of periodic forcing) typical of nonlinear systems. Consider the following system of nonlinear equations:
$$\begin{aligned} \dot{x}_1 &= x_2 + \alpha x_1(\beta^2 - x_1^2 - x_2^2) ; \\ \dot{x}_2 &= -x_1 + \alpha x_2(\beta^2 - x_1^2 - x_2^2) . \end{aligned}$$
Introduce polar coordinates in the form
$$\begin{aligned} r &= \sqrt{x_1^2 + x_2^2} ; \\ \phi &= \tan^{-1}\left(\frac{x_2}{x_1}\right) . \end{aligned}$$
Then, the system is written as
$$\begin{aligned} \dot{r} &= \alpha r(\beta^2 - r^2) ; \\ \dot{\phi} &= -1 . \end{aligned}$$

We can see that the system admits the steady state solution,

$$r^2 = \beta^2 \quad \text{or} \quad x_1^2 + x_2^2 = \beta^2 .$$

This represents a periodic solution which — unlike the simple harmonic oscillator case where there is a continuous family of periodic solutions depending on the initial conditions — is isolated. Such a periodic solution is called a limit cycle.

As another example of a limit cycle, consider the so–called Van der Pol equation, which models a spring–mass–damper system with nonlinear damping,

$$m\ddot{x} - b(1 - x^2)\dot{x} + kx = 0 .$$

For small $x$ it becomes linear,

$$m\ddot{x} - b\dot{x} + kx = 0 .$$

The equilibrium point is $x = 0$, which is clearly unstable due to negative damping. Therefore, solutions which start in the neighborhood of $x = 0$ must move away from it. On the other hand, for large values of $x$ the damping becomes positive. Therefore, solutions that start far away from $x = 0$ must move towards the origin. Since solution curves cannot cross each other (such crossiong would violate uniqueness of solutions of ordinary differential equations), there must be a limit cycle in between which both sets of solution curves approach asymptotically. **MATLAB** has a nice differential equations **demo** which illustrates the Van der Pol limit cycle.

5. Types of behavior: The various types of possible behavior in nonlinear systems depend heavily on system dimensionality. Thus:

   - First–order systems may exhibit only equilibrium points.
   - Second–order systems may exhibit either equilibrium points or limit cycles.
   - Higher–order systems may exhibit equilibrium points, limit cycles, and a plethora of other more complex response patterns.

   Forced and/or discrete systems can be considerably more complicated.

6. Frequency entrainment: If a periodic force of frequency $\omega$ is applied to a system capable of exhibiting a limit cycle of frequency $\omega_0$, we have the phenomenon of beats. As the difference between the two decreases, the beat frequency also decreases and, for a linear system it is zero only if $\omega = \omega_0$. In a self–excited nonlinear system, however, it is found that the frequency $\omega_0$ of the limit cycle falls in synchronization with, or is entrained by, the forcing frequency $\omega$ within a certain band of frequencies.

7. Types of nonlinearities: Some inherent nonlinearities of particular significance to hydraulic systems are shown in Figure 63. Such nonlinearities can be either part of the physical structure of the system or can be ad–hoc introduced through software commands.

2. Describing Functions

There are a few tools that can be used to predict the existence, magnitude, and stability of limit cycles, namely,

- numerical integrations,
- continuation methods,
- perturbation methods,
- describing function analysis.

Numerical integrations are easy to apply but the can only be used to confirm rather than predict possible behavior, especially when a large number of variables and initial conditions are present. Continuation methods require some initial approximation of the limit cycle for a given set of parameters, while perturbation methods are best applied to system with smooth nonlinearities, unlike the ones depicted in Figure 63. Describing function analysis is an approximate method that is best suited to the discontinuous nonlinearities common in fluid power systems.

Suppose that the input to a nonlinear element is sinusoidal. The output will be periodic and suppose that only the component with the same frequency as the input (the fundamental harmonic component) is significant. The complex quantity

$$G_d = \frac{C_1}{M} \angle \phi_1 \; ;$$

where

| | | |
|---|---|---|
| $M$ | = | amplitude of input sinusoid |
| $C_1$ | = | amplitude of fundamental harmonic component of output |
| $\phi_1$ | = | phase shift of fundamental harmonic component of output |

is called the describing function $G_d$.

3. Computation of Describing Functions

For a sinusoidal input

$$m(t) = M \sin \omega t$$

to the nonlinear element, the output $c(t)$ may be expressed in Fourier series as follows:

$$\begin{aligned} c(t) &= A_0 + \sum_{n=1}^{\infty} (A_n \cos n\omega t + B_n \sin n\omega t) \\ &= A_0 + \sum_{n=1}^{\infty} (C_n \sin(n\omega t + \phi_n) \; ; \end{aligned}$$

where

$$
\begin{aligned}
A_n &= \frac{1}{\pi}\int_0^{2\pi} c(t)\cos(n\omega t)\,d(\omega t);\\
B_n &= \frac{1}{\pi}\int_0^{2\pi} c(t)\sin(n\omega t)\,d(\omega t);\\
C_n &= \sqrt{A_n^2 + B_n^2};\\
\phi_n &= \tan^{-1}\left(\frac{A_n}{B_n}\right):
\end{aligned}
$$

If the nonlinearity is symmetric, then $A_0 = 0$. The fundamental harmonic component of the output is

$$
\begin{aligned}
c_1(t) &= A_1\cos\omega t + B_1\sin\omega t\\
&= C_1\sin(\omega t + \phi_1):
\end{aligned}
$$

The describing function is then given by,

$$
G_d = \frac{C_1}{M}\langle\phi_1\rangle = \frac{\sqrt{A_1^2 + B_1^2}}{M}\angle\left(\tan^{-1}\left(\frac{A_1}{B_1}\right)\right):
$$

As an example, consider the saturation nonlinearity of Figure 64. A Fourier calculation of the output waveform for a sinusoidal input gives the following describing function

$$
G_d = \frac{2}{\pi}\left[\sin^{-1}\left(\frac{S}{M}\right) + \frac{S}{M}\sqrt{1 - \left(\frac{S}{M}\right)^2}\,\right]:
$$

For a stauration function of slope k the term $2/\pi$ in front of the above expression becomes $2k/\pi$. Also, this expression is true for $S < M$. For $S > M$, the input signal does not feel the effects of the saturation and it behaves just like a linear unity gain; i.e., $G_d = 1$ for $S > M$. A plot of the saturation describing function $G_d$ versus the dimensionless ratio $S/M$ is shown in Figure 65. A very useful general property for calculating describing functions is:

> The describing function of the sum of two elements is the sum of the individual describing functions.

4. Describing Function Analysis

Consider the closed-loop feedback system of Figure 66 containing a linear element with transfer function G and a nonlinear element with describing function $G_d$. If the higher harmonics are sufficiently attenuated, the describing function $G_d$ can be treated as a complex gain. Then, the closed loop frequency response is

$$
\frac{C(j\omega)}{R(j\omega)} = \frac{G_dG(j\omega)}{1 + G_dG(j\omega)}:
$$

The characteristic equation is

$$
1 + G_dG(j\omega) = 0;
$$

or

$$G(j\omega) = -\frac{1}{G_d(M)}:$$

If this equation is satisfied, then the system will exhibit a limit cycle with frequency $\omega$ and amplitude $M$ found from the intersection of $G(j\omega)$ and $-1/G_d(M)$ graphs.

5. Stability of Limit Cycles

To assess the stability of these limit cycles, we have to recognize the similarity between the above and the Nyquist criterion for linear systems. For example, consider the case shown in Figure 67. We see that we have two limit cycles with characteristics $(M_A; \omega_A)$ and $(M_B; \omega_B)$ with $M_A < M_B$. Consider the intersection $A$ of the $G(j\omega)$ and $-1/G_d(M_A)$ loci and assume a small decrease in amplitude $M_A$. The representative point on the $-1/G_d$ locus will move to a new point, $D$. This point is not encircled by the $G(j\omega)$ locus, the system will move further and further away from the intersection and the oscillations will eventually stop. Therefore, point $A$ possesses divergent characteristics and it corresponds to an unstable limit cycle. By a similar argument we can see that point $B$ possesses convergent characteristics and it corresponds to a stable limit cycle. Indeed, if the amplitude of the limit cycle is decreased so that the system moves to point $F$ we can see that the new point is encircled by the $G(j\omega)$ locus, the oscillations will grow, the system will tend to return to the original intersection $B$ and the oscillations are stable. As a summary, we can conclude that in general: The limit cycle is predicted to be stable or unstable according as the locus of $-1/G_d$ crosses the locus of $G$ (the Nyquist plot) from right to left or from left to right, respectively, as $M$ increases, viewed along the direction of increasing $\omega$. This criterion is illustrated by the sketch of Figure 68.

6. Example: Saturation

Consider a linear system with the saturation nonlinearity shown in Figure 64. Suppose that the Nyquist diagram for the linear element encloses the $-1$ point, so that the linear system is unstable. If there were no saturation, this means that oscillations with ever–increasing amplitude would develop. To analyze the effect of saturation let us superimpose the graph of the describing function of the saturation nonlinearity onto the Nyquist diagram, as shown in Figure 69. We can see that the effect of the saturation (i.e., limit on actuator stroke) is to generate a stable limit cycle at the intersection point and thus prevent the motions from becoming arbitrarily large. If the gain of the transfer function is decreased so that the locus of $-1/G_d$ does not intersect that of $G$, the system becomes stable and any oscillations that may develop will eventually die out. No limit cycle (self sustained oscillation) will exist at steady state.

As another example consider the effects of saturation on a conditionally stable system as shown in Figure 70. The linear system is here stable since the polar plot avoids the $-1$ point. In this case we can see that two limit cycles are created one at $P_1$ and another one at $P_2$. The limit cycle at $P_1$ is unstable, whereas the limit cycle at $P_2$ is stable. Therefore, if the system amplitude exceeds this value, for example during transient response, self–sustained oscillations with amplitude corresponding to $P_2$ will develop. In this case even though the origin is stable, the effect of the saturation is to limit the origin's domain of attraction.

System response will converge to zero as long as the initial transient does not exceed $P_1$.

7. Example: Deadband

A deadband nonlinearity (Figure 71) can result from Coulomb friction and from overlap of valve ports in hydraulic systems. The linear gain of the deadband is normalized to one and any gain present would be considered as part of the linear portion of the loop. Analysis of the output waveform gives the following describing function

$$G_d = \frac{2}{\pi}\left[\frac{\pi}{2} - \sin^{-1}\left(\frac{D}{M}\right) - \frac{D}{M}\sqrt{1-\left(\frac{D}{M}\right)^2}\,\right];$$

which is plotted in Figure 72.

We note that $-1/G_d$ is a large negative real number for small inputs to the deadband element and approaches $-1$ for large inputs. Suppose the polar plot is as shown in Figure 73. The linear system with this Nyquist plot would be unstable. The limit cycle at the intersection point is also unstable. This means that the system will actually be stable for small inputs to the deadband (i.e., as long as the intersection point is not crossed over). If it seems peculiar that an unstable linear system may become stable with the addition of a nonlinear element, this is due to the fact that the actual system including the deadband has very small gain at the origin. In this case, since the deadband generates an unstable limit cycle, unbounded oscillations will occur if the input to the deadband is large enough. This is why deadbands are quite undesireble from the stability point of view. In any practical system, however, the deadband will saturate and the oscillations will become bounded. This case is treated next.

8. Example: Nonlinear Gain Characteristics

The describing function of the general nonlinear gain characteristic in Figure 74 is,

$$\begin{aligned} G_d \;=\; & k_3 + \frac{2}{\pi}(k_1-k_2)\left[\sin^{-1}\left(\frac{D}{M}\right) + \frac{D}{M}\sqrt{1-\left(\frac{D}{M}\right)^2}\,\right] \\ & + \frac{2}{\pi}(k_2-k_3)\left[\sin^{-1}\left(\frac{S}{M}\right) + \frac{S}{M}\sqrt{1-\left(\frac{S}{M}\right)^2}\,\right]. \end{aligned}$$

The describing functions for saturation and deadband can be obtained from this expression by letting appropriate quantities be zero. With so many parameters involved, it is better to look at a particular case. Of interest is a combination of saturation and deadband (Figure 75). In this case $k_1 = k_3 = 0$ and $k_2 = 1$ and the describing function is plotted in Figure 76. Note that the "gain" is small for small inputs, increases to a maximum, then decreases as the input amplitude $M$ increases. Thus, the quantity $-1/G_d$ starts at $-1$ for small inputs, decreases to a minimum, then again approaches $-1$ as the input becomes very large. The $-1/G_d$ locus and a polar plot of a linearly unstable system are shown in Figure 77. For the intersections shown, point $P_1$ is an unstable limit cycle and $P_2$ is a stable limit cycle. Note that this system is stable for small inputs not exceeding $P_1$, but once the input amplitude becomes greater than at point $P_2$, oscillations will build up to a limit cycle at $P_2$. The $-1/G_d$

locus has a minimum which approaches but never exceeds the $-1$ point. Thus, a system having this characteristic and designed so that the polar plot does not encircle the $-1$ point would be stable. However, it is possible for the system to be stable even if the $-1$ point is encircled because of the minimum of the $-1/G_d$ locus.

9. Backlash and Hysteresis

Backlash and hysteresis nonlinearities are multivalued. With backlash, the input must be moved by a certain amount before any motion of the output occurs. Similarly upon reversal. Generally speaking, backlash can pose a serious threat to the stability of a loop. Dither is a widely used method of removing backlash. Its is very effective where the backlash is caused by friction. Dither is a high frequency signal of constant amplitude and frequency which is added to the control signal at the input to the nonlinearity and has the effect of making the element appear linear. However, dither cannot be used in certain cases such as gear backlash because it is difficult to inject, causes wear, and shows in the output.

Hysteresis nonlinearities constitute a nuisance but not a serious threat to stability. The most noticeable attribute of elements with hysteresis nonlinearity is an amount of phase lag at low frequencies.

10. Comments

The describing function analysis is an extension of linear techniques to the study of nonlinear systems. Typical applications are to systems with few nonlinearities. The analysis is only approximate: there are instances where the describing function analysis predicts the existence of limit cycles but the actual system exhibits none, and other instances where the situation is reversed.

It is more accurate to state that the describing function analysis predicts the likelihood of limit cycles. The system may exhibit a periodic solution with amplitude and frequency close to the predicted ones. Final response has to be verified by numerical integrations.

11. A Counter–example: Van der Pol's Equation

Once more, consider Van der Pol's equation

$$\ddot{y} + \epsilon(3y^2 - 1)\dot{y} + y = 0 .$$

In order to represent this in a "block diagram" form including an appropriate nonlinear element, we write it as,

$$\begin{aligned}
&\ddot{y} - \epsilon\dot{y} + y = -3\epsilon y^2\dot{y} \quad \text{or}\\
&\ddot{y} - \epsilon\dot{y} + y = -\epsilon\frac{d}{dt}y^3 \quad \text{or}\\
&(s^2 - \epsilon s + 1)y = \epsilon s(-y^3) \quad \text{or}\\
&\frac{y}{u^3} = \frac{\epsilon s}{s^2 - \epsilon s + 1} .
\end{aligned}$$

Therefore, in feedback form,

$$G(s) = \frac{\epsilon s}{s^2 - \epsilon s + 1} ;$$

with the nonlinearity $f(u) = u^3$, and zero reference input, so that $u = -y$, see Figure 78. For the cubic nonlinearity,

$$G_d = \frac{3M^2}{4}:$$

In order to predict the limit cycle we have to solve

$$G(j\omega) = -\frac{1}{G_d(M)};$$

or

$$\frac{\varepsilon j\omega}{-\omega^2 - \varepsilon j\omega + 1} = -\frac{4}{3M^2};$$

or

$$4(\omega^2 - 1) + j(4 - 3M^2)\varepsilon\omega = 0:$$

Therefore, the frequency of the limit cycle is predicted at

$$\omega = 1 \quad (\text{period } 2\pi);$$

and its amplitude at

$$M = \frac{2}{\sqrt{3}}:$$

The graphical construction easily shows that this limit cycle is stable.

Now although Van der Pol's equation cannot be solved analytically, it is possible to obtain asymptotically exact expressions for the limit cycle parameters as $\varepsilon$ approaches zero or infinity. In the small parameter limit ($\varepsilon \to 0$), the equation becomes that of a simple harmonic oscillator with unit angular frequency, coinciding with the prediction of the describing function method. In the large parameter limit ($\varepsilon \to \infty$), a perturbation analysis predicts period $1.614\varepsilon$, instead of fixed $2\pi$. In order to understand why the method fails in this case, take a closer look at the frequency response of the linear component:

$$G(j\omega) = \left[ -1 + \frac{j}{\varepsilon}\left(\omega - \frac{1}{\omega}\right) \right]^{-1}:$$

It is clear that, as $\varepsilon$ increases, so does the range of $\omega$ over which $G(j\omega) \approx -1$. This means that in the limit of infinite $\varepsilon$ we obtain an "all-pass" filter, and hence the harmonic content of the limit cycle becomes such that the predominant response is no longer simply sinusoidal, and the describing function approximation cannot be expected to be valid any more.

These notes utilize material mainly from the following two sources:

1. Healey, A. J., Hydraulic Components.

2. Merrit, H. E., Hydraulic Control Systems.